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Solution of Homogeneous and Inhomogeneous Linear differential Equations Systems Using Matrices.

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Abstract:
The study addressed the solution of homogeneous and inhomogeneous linear differential equation systems using matrices. The systems of linear ordinary differential equations have important applications,possesspowerful tools in study and solvevarious problems in natural sciences, technology and social sciences. The matrices technique has been used in solving homogeneous and inhomogeneous linear differential equation systems by generating, eigevalues and eigenvectors of coefficient matrices along with many theories, definitions have been introduced in order to accelerate the process.
It has been successfully implemented and found that, the method gives a typical results comparing with other analytical methods.However, the drawback of the scheme is the generating of eigenvalues, eigenvector and calculating matrices inverseare tedious and sophisticate when the order of coefficients matrices more than3×3.

KEY WORD: Systems of linear differential equations, homogeneous, inhomogeneous, Eigen values, Eigenvectors, Applications

1.0 Introduction
The systems of linear ordinary differential equations have important applications,possess powerful tools in study and solvevarious problems in natural sciences, technology;these are extensively employed in mechanics, astronomy, physics, chemistry and biology problems. The reason for this is the fact that objective laws governing certain phenomena (processes) can be written as ordinary differential equations, so that the equations themselves are a quantitative expression of these laws. For instance, Newton's laws of mechanics make it possible to reduce the description of the motion of mass points or solid bodies to solving ordinary differential equations. The computation of radiotechnical circuits or satellite trajectories, studies of the stability of a plane in flight, and explaining the course of chemical reactions are all carried out by studying and solving ordinary differential equations.
The principal tasks of the theories of ordinary differential equations of linear systems concernthe study of solutions of such equations. However, the meanings of such a study of solutions have been understood in various ways at different times. The original trend is to carry out the integration of equations in quadratic to obtain a closed formula yielding (in explicit, implicit or parametric form. The linear systems of ordinary differential equations (ODEs) arise in many contexts of mathematics, sciences to descript the change or the rate of change in many dynamics phenomena. Often, quantities are defined as the rate of change of other quantities (for example, derivatives of displacement with respect to time), or gradients of quantities, which is how they enter differential equations.

It's worth mentioning, there are many approaches and techniques to solve ODEs of linear systems which describe several phenomena in our daily life,and one of these methods is the solution of systems of linear differential equations for homogenous and inhomogeneous form and different order. Many of the applications involve the use of eigenvalues and eigenvectors in the process of transforming a given matrix into a diagonal matrix.

Firstly, the study implemented the matrices solution idea using eigenvalues and eigenvectors transforming. However the study shows how these processesare invaluable in solving coupled differential equations of both first order and second order in homogenous form. Secondly the idea extended to solve inhomogeneous systems for linear differential equations form.

1.1 Definition, Theorems and Properties of Eigen values and Eigenvectors

The Eigen values and eigenvectors are invaluable tools in solving the linear systems of differential equations which well describes many phenomena in natural and social sciences and gives good deep insight into sophisticate mathematical problems in all branches of sciences, so in this section we introduced, definitions, theorems and properties of eigenvalues and eigenvector in order to simplify the techniques and posses the scientists with good background.

Definition (1.1): Let A be an $n \times n$matrix.If there is a number $\lambda \epsilon \mathbb{R}$ and an n-vector $x \neq 0$ such that $Ax = \lambda x$, then we say that λ is an eigenvalue forA, and x is called an eigenvector for Awith eigenvalueλ.

Definition (1.2): The set of all eigenvectors of A for a given eigenvalue λ is called an eigenspace, and it is written Eλ(A).

Definition (1.3): let A be ann $\times$ n matrix. A scalar λ is called an eigenvalue of A if there is an a non zero vectorv $\neq 0$, called an eigenvector, such that

$Av = \lambda v$ (1.1)

In the other words, the matrix A stretches the eigenvector, v by an amount specified by the eigenvalue. The requirement that the eigenvector v be nonzero is important, since $v = 0$ is a trivial solution to the eigen equation (1.1) for any, scalarλ. Moreover, as far as solving linear ordinary differential equations goes, the zero vector $v = 0$ gives $u(t) = 0$, which is certainly a solution.

The Eigen value equation (1.1) is a system of linear equations for the entries of the eigenvector v − provided that the Eigen value.

Let $(A - \lambda I)v = 0$ (1.2)

Where I is the identity matrix of the correct size, for the given λ, equation (1.2) lead us to homogenous linear system for v.

The homogenous linear system has anon zero solution $v \neq 0$ if and only if its coefficient matrix

$(A - \lambda I) = 0$is singular.

Definition (1.4): Let A and B are square matrices of the same order. Then, A is said to be similar to B if there exists an invertible matrix P such that $P^{-1}AP = B$. The matrix P is called a similarity transformation matrix. Note that if A is similar to B, then B is similar to A and the two matrices are called similar matrices.

Definition (1.5) The matrix $A$is diagonalizable if it is similar to a diagonal matrix, that is, there is an invertible matrix $P$and diagonal matrix Dsuch that $P^{-1}AP = D$. In this case we say that $P$is a diagonalizing matrix for A or that PdiagonalizesA. We can be more specific about when a matrix is diagonalizable. As a first step, notice that the calculations that we began can easily be written in terms of an$n \times n$matrix instead of a 3×3matrix. What these calculations prove is the following basic fact.

Proposition (1.1): A scalar $\lambda$is an eigenvalue of the matrix A if and only if λ is a solution to the characteristic equation.

$\det(A - \lambda I) = 0$ (1.3)

Proposition (1.2): If A is a real matrix with a complex eigenvalue $\lambda = u + iv$ and corresponding complex eigenvector $v = x + iy$, then the complex conjugate $\lambda^{-} = u - iv$is also an eigenvalue with complex conjugate eigenvector $v^{-} = x - iy$

Proof:

Let the complex conjugate of the eigenvalue equation (1.1).

$$\overline{\mathrm{A}}\overline{\mathrm{v}} = \overline{\mathrm{Av}} = \overline{\lambda\mathrm{v}} = \bar{\lambda}\overline{\mathrm{v}}$$

Using the fact that a real matrix is unaffected by conjugate, so $\bar{A} = A$, we conclude $\bar{\lambda}\bar{\mathrm{v}} = \mathrm{A}\bar{\mathrm{v}}$, which is the equation for the eigenvalue $\bar{\lambda}$and eigenvector $\bar{v}$. As a sequence, when dealing withreal matrices, we only need to compute the eigenvector for one of each complex conjugate pair of eigenvalues.

Proposition (1.3): A matrix A is singular if and only if 0 is an eigenvalue.

Proof:

By definition, 0 is an eigenvalue of A if and only if there is a nonzero solution to the eigenvector equation $Av = 0v$. Thus, 0 is an eigenvector of A if and only if it has a nonzero vector in its kernel, $kerA \neq \{0\}$, and hence $A$is necessarily singular

Properties (1.4): IfA is an $n \times n$ matrix, then the characteristic polynomial is

$$P_A(\lambda) = \det(A - \lambda I) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} \dots \dots \dots \dots c_0\lambda + c_0 \qquad (1.4)$$

The value $P_A(\lambda)$ is a polynomial of degree n is a sequence of the general determinant formula and every term is prescribed by a permutation π of the rows of the matrix and equal plus or minus a product of $n$distinct matrix entries including one from each row and one of each column. The term corresponding to the identity permutation is obtained by multiplying the diagonal entries together, which, in this case, is

$$(a_{11} - \lambda)(a_{22} - \lambda) \dots \dots (a_{nn} - \lambda) = (-1)^n\lambda^n + (-1)^{n-1}(a_{11} + a_{22} + \cdots a_{nn})\lambda^{n-1} + \cdots \qquad (1.5)$$

All of the other terms have at most $n - 2$ diagonal factors$a_{ii} - \lambda$, and so are polynomials of degree $\leq n - 2$ in λ. Thus the equation (1.5) is the only summand containing the monomials $\lambda^n$and λ^{n-1}, and so their respective coefficients are

$$c_n = (-1)^n, c_{n-1} = (-1)^{n-1}(a_{11} + a_{22} + \cdots a_{nn}) = (-1)^{n-1}\mathrm{trA} \qquad (1.6)$$

WheretrA, the sum of its diagonal entries, is called the $trace$ of the matrixA. The other coefficients $c_{n-2} \dots c_1, c_0$in equation (1.5) are more complicated combinations of the entries of A. However, put $\lambda = 0$, implies that $P_A(0) = \det(A) = c_0$, so that the constant term in the characteristic polynomial equals the determinant of the matrix. In particular, if $\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an 2×2 matrix, its characteristic polynomial has the explicit form

$$P_A(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc) = \lambda^2 - (trA)\lambda + (detA) \; (1.7)$$

Theorem (1.5): A scalar λ is an eigenvalue of the $n \times n$ matrix A if and only if the matrix $A - \lambda I$ is singular, i.e. of rank $< n$. The corresponding eigenvectors are non zero solutions to the eigenvalue equation $(A - \lambda I)v = 0$

Theorem (1.6): Let $D = diag\{\lambda_1, \lambda_2, \dots, \lambda_n\}$be a diagonal matrix. Then $D^k = diag\{\lambda_1^k, \lambda_2^k, \lambda_3^k, \dots \dots \dots \dots \lambda_n^k\}$, where k is positive integer. Let’s now consider a 3 × 3 matrix A.If we could find three linearly independent eigenvectors v_1, v_2 and $v_3$corresponding to the eigenvaluesλ_1, λ_2 andλ_3, we would have $Av_1 = \lambda_1 v_1, Av_2 = \lambda_2 v_2$, and $Av_3 = \lambda_3 v_3$

In matrix form, we have $[\boldsymbol{v1}, \boldsymbol{v2}, \boldsymbol{v3}] = [\boldsymbol{v1}, \boldsymbol{v2}, \boldsymbol{v3}]\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = [v1, v2, v3]diag[\lambda 1, \lambda 2, \lambda 3]$

Now, set$P = [v_1, v_2, v_3]$and $D = diag[\lambda_1, \lambda_2, \lambda_3]$.Then P is invertible since the columns of P are linearly independent. Multiplying both sides of $AP = PD$by $P^{-1}$to the left, we get $P^{-1}\mathrm{AP} = \mathrm{D}$. this is a beautiful equation, because it makes the powers of A simple to understand. The procedure we just went through is reversible as well. In other words, if $P$is an invertible matrix such that$P^{-1}\mathrm{AP} = \mathrm{D}$.then we deduce that $AP = PD$and conclude that the columns of $P$are linearly independent eigenvectors of A. We make the following definition and follow it with a simple but key theorem relating similar matrices.

Theorem (1.7): Let $v_1, v_2, \dots, v_k$be a set of eigenvectors of the matrix Asuch that corresponding eigenvalues are all distinct. Then, the set of vectors $\{v_1, v_2, \dots, v_k\}$is linearly independent.

Proof: Suppose the set is linearly dependent. Discard redundant vectors until we have a smallest linearly dependent subset such as $v_1, v_2, \dots, v_m$ with v_i belonging to λ_i. All the vectors have nonzero coefficients in a linear combination that sums to zero, forwe could discard the ones that have zero coefficient in the linear combination and still have a linearly dependent set. So thereis some linear combination of the form

$c_1v_1 + c_2v_2 + \dots + c_mv_m = 0$ (i)

with each $c_j \neq 0$ and v_j belonging to the eigenvalue λ_j. Multiply equation (i) by λ_1 to obtain the equation

$c_1\lambda_1v_1 + c_2\lambda_1v_2 + \dots + c_m\lambda_1v_m = 0$(ii)

Next, multiply equation (1) on the left by A to obtain

$0 = A(c_1v_1 + c_2v_2 + \dots + c_mv_m) = c_1Av_1 + c_2Av_2 + \dots + c_mAv_m$

That is $c_1\lambda_1v_1 + c_2\lambda_2v_2 + \dots + c_k\lambda_mv_m = 0$(iii)

Now subtract equation(3) from equation(2) to obtain

$0v_1 + c_2(\lambda_1 - \lambda_2)v_2 + \dots + c_k(\lambda_1 - \lambda_m)v_m = 0$

This is a new nontrivial linear combination ($since\ c_2(\lambda_1 - \lambda_2) \neq 0$) of fewer terms, that contradicts our choice of $v_1, v_2, \dots, v_k$. It follows that the original set of vectors must be linearly independent.

Theorem (1.8): If A is an $n \times n$ symmetric matrix, then the following properties are true.

1- A is diagonalizable.

2. All eigenvalues of A are real.

3. If λ is an eigenvalue of A with algebraic multiplicity k, then λ has k linearly independent eigenvectors. That is, the eigenspace of λ has dimension k.

Theorem (1.9): If the coefficient matrix A is complete, then the general solution to the linear iterative systems $x^{(k+1)} = Ax^{(k)}$. is given by

$x^{(k)} = c_1\lambda_1^k v_1 + c_2\lambda_2^k v_2 + \cdots c_n\lambda_n^k v_n$

2. SOLVING SYSTEMS OF FIRST ORDER DIFFERENTIAL EQUATIONS

Systems of first order linear ordinary differential equations arise in many areas of mathematics and engineering, for example in control theory and in the analysis of electrical circuits. In each case the basicunknowns are each a function of the time variable t. A number of techniques have been developedto solve such systems of equations; for example the Laplace transform. Here we shall use eigenvaluesand eigenvectors to obtain the solution. Our first step will be to recast the system of ordinary differentialequations in the matrix form $X' = AX$ where A is an $n \times n$ coefficient matrix of constants, X is the $n \times 1$ column vector of unknown functions and X' is the $n \times 1$ column vector containing thederivatives of the unknowns. The main step will be to use the modal matrix of A to diagonalisethe system of differential equations. This process will transform $X' = AX$ into the form $Y' = DY$ where D is a diagonal matrix. We shall find that this new diagonal system of differential equationscan be easily solved. This special solution will allow us to obtain the solution of the original system.

Consider a system of ordinary first order differential equations of the form

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots a_{1n}x_n + b_1 \\
x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots a_{2n}x_n + b_2 \\
x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots a_{3n}x_n + b_3 \\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots \\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots a_{nn}x_n + b_n \qquad (3.1)
\end{aligned}
$$

Where, $a_{ij} \in \mathbb{R}$.

If $b_1 = b_2 = \cdots = b_n = 0$, the systems are called homogenous and if not the systems are called inhomogeneous we can form equation (1-1) as $X' = A(t)X + B(t)$

Where $A(t): I \to \mathbb{R}^{n\times n}$, $B(t): I \to \mathbb{R}^n$ and I being an Open interval in $\mathbb{R}$.

If $A(t)$ and B(t) are continuous in t then for any $t_0 \epsilon I$ and $x_0 \in \mathbb{R}^n$, then IVP

$$
\begin{cases} x' = A(t)x + B(t) \\ \quad x(t_0) = x_0 \end{cases} \qquad (3.2)
$$

The systems (3.1) and (3.2) has a unique solution defined on the interval I

3.1 Space of Solutions of homogeneous systems

A linearODE is called homogeneous if $B(t) \equiv 0$ and inhomogeneous otherwise. If $x' = A(t)x$ be a homogeneous equation the $\mathcal{A}$ the set of all solution of ODE in (1-2), this idea produce from the following theorem

Theorem (3.1): $\check{\mathcal{A}}$ is a linear space of dim$\check{\mathcal{A}} = n$. Consequently if $x_1, x_2, \dots \dots \dots x_n$ are n linearly independent solution to $x' = A(t)x$, then the general solution has the form

$$x(t) = c_1x_1(t) + c_2x_2(t) + \cdots c_nx_n(t)$$

Where $c_1, c_2 \dots \dots \dots \dots \dots \dots c_n$ are arbitrary constant

3.1.1 The Method Implementations

(i) Consider the following first order linear differential equations system.

$$y' = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} y$$

Find the general solution for the above system

Solution:

To find the eigenvalues $\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{bmatrix} y = \lambda^3 - 2\lambda^2 - 5\lambda + 6$

$$\lambda = 1, 3, -2$$

When$\lambda = 1$, we will get $A - \lambda I \Longrightarrow \begin{bmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$

Therefore $-v_2 + v_3 = 0,\ 3v_1 + v_2 - v_3 = 0, 2v_1 + v_2 - 2v_3 = 0,$

Thus $v_1 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$

When $\lambda = 3$, then an eigenvector can be form as $v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Also when $\lambda = -2$ then an eigenvector can be form as, $v_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Since v_1, v_2, v_3 are eigenvectors corresponding to distinct eigenvalues, therefore they are linearly independent, and hence the general solution to the problem can be written as

$$y(t) = c_1 \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{-2t}$$

(ii) Find the anlytical solution for System of First Order Linear Differential Equations

Let the first order linear differential equations of the form

$$\mathrm{x}_1' = 3\mathrm{x}_1$$
$$x_2' = 5x_2$$
$$x_3' = -2x_3$$

Solution:

From the principle of the general form of the solution is

$x' = kx$is $x = c_ie^{kt}$, therefore the solutions of the system is

$$x_1 = c_1e^{3t}$$

$x_2 = c_2e^{5t}$

$x_3 = c_3e^{-2t}$

The matrix form of the system of linear differential equations for above example is

$X' = AX$, where A is square matrix such that

$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, thus A is diagonal matrix, then the solutions of the system $X' = AX$ can be obtained as above solutions

If A is not diagonal matrix, so this need more procedures, first we try to find the matrix P that diagonalizesA such the change of variables arise

$\mathrm{X} = \mathrm{PH}$ and $\mathrm{X}' = \mathrm{PH}'$produces

$$\mathrm{PH}' = \mathrm{X}' = AX = A\mathrm{PH} \Rightarrow \mathrm{H}' = \mathrm{P}^{-1}A\mathrm{PH}$$

Where $\mathrm{P}^{-1}A\mathrm{P}$ is a diagonal matrix.

(iii) Solve the following the system of linear differential equations
$x_1' = 3x_1 + 2x_2$, $x_2' = 6x_1 \pm x_2$
Solution:
To find the matrix H that diagonalizes $A = \begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix}$, the eigenvalues of A are $\lambda_1 = -3, \lambda_2 = 5$ and the corresponding eigenvector $h_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $h_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively, then $H = [h_{1,} h_2] = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$

$H' = \begin{bmatrix} \frac{1}{4} & \frac{-1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}$. Therefore to diagonalizes A as $H'AH = . \begin{bmatrix} \frac{1}{4} & \frac{-1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} . \begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix} . \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$

The system $X' = P^{-1}APX$ follow the form $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_1' = -3x_1, x_2' = 5x_2$
The solution of this system of equations is
$x_1 = c_1 e^{-3t}$ and $x_2 = c_2 e^{5t}$
To return to the original variables x_1 and x_2 use the substitution
$X_1 = x_1 + x_2 = c_1 e^{-3t} + c_2 e^{5t}$ and $X_2 = -3x_1 + x_2 = -3c_1 e^{-3t} + c_2 e^{5t}$
If A has eigenvalues with multiplicity greater than 1 or if Ahas complex eigenvalues, then the technique for solving the system must be modified.

(iv)Solution of linear systems of the form $\mathbf{x}^{(\mathbf{k+1})} = \mathbf{A}\mathbf{x}^{(\mathbf{k})}$ containing complex roots.

Let $A = \begin{pmatrix} -3 & 1 & 6 \\ 1 & -1 & -2 \\ -1 & -1 & 0 \end{pmatrix}$ be the coefficient matrix for three dimensional iterative systems $x^{(k+1)} = Ax^{(k)}$.
If the eigenvalues and corresponding eigenvectors are: $\lambda_1 = -2, \lambda_2 = -1 + i, \lambda_3 = -1 - i$
$v_1 = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 - i \\ -1 \\ 1 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 2 + i \\ -1 \\ 1 \end{pmatrix}$
Therefore , according to theorem (1.9) the general complex solution to the iterative differential equations of linear systems is

$$x^{(k)} = b_1(-2)^k \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} + b_2(-1+i)^k \begin{pmatrix} 2 - i \\ -1 \\ 1 \end{pmatrix} + b_3(-1-i)^k \begin{pmatrix} 2 + i \\ -1 \\ 1 \end{pmatrix}$$

Where b_1, b_2, b_3 are arbitrary complex scalars
We notce that the solution has real and imaginary parts, so in the case of systems of differential equations, we break up any complex solution into its real and imaginary parts , each of which constitutes a real solution. by writing , $\lambda_2 = -1 + i = \sqrt{2} e^{\frac{3\pi i}{4}}$ and hence

$$(-1+i)^k \begin{pmatrix} 2 - i \\ -1 \\ 1 \end{pmatrix} = 2^{k/2} e^{3k\pi i/4} = 2^{k/2} (\cos\frac{3}{4}k\pi + i \sin\frac{3}{4}k\pi)$$

Hence the complex of complex solution

$$(-1+i)^k \begin{pmatrix} 2 - i \\ -1 \\ 1 \end{pmatrix} = 2^{k/2} \begin{pmatrix} 2\cos\frac{3}{4}k\pi + \sin\frac{3}{4}k\pi \\ -\cos\frac{3}{4}k\pi \\ \cos\frac{3}{4}k\pi \end{pmatrix} + i2^{k/2} \begin{pmatrix} 2\sin\frac{3}{4}k\pi - \cos\frac{3}{4}k\pi \\ -\sin\frac{3}{4}k\pi \\ \sin\frac{3}{4}k\pi \end{pmatrix}$$

Which is indicating that, the solution is a combination of two real solutions? Also the complex conjugate eigenvalue $\lambda_3 = -1 - i$ leads to complex conjugate solution, The general real solution $x^{(k)}$ to the system can be written as a linear combination of the three independent real solutions

$$x^{(k)} = c_1(-2)^k \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} + c_2 2^{k/2} \begin{pmatrix} 2\cos\frac{3}{4}k\pi + \sin\frac{3}{4}k\pi \\ -\cos\frac{3}{4}k\pi \\ \cos\frac{3}{4}k\pi \end{pmatrix} + c_3 2^{k/2} \begin{pmatrix} 2\sin\frac{3}{4}k\pi - \cos\frac{3}{4}k\pi \\ -\sin\frac{3}{4}k\pi \\ \sin\frac{3}{4}k\pi \end{pmatrix}$$

Where c_1, c_2, c_3 are arbitrary real scalars and $x^{(k)}$ is the solutions of $x^{(k+1)} = Ax^{(k)}$.

3.2 Space of Solutions of inhomogeneous systems

Consider the inhomogeneous linear ODE

$x' = A(t)x + B(t)$ (3.3)

Where $A(t): I \to \mathbb{R}^{n\times n}$, $B(t): I \to \mathbb{R}^n$ are continuous mapping on open interval $I \subset \mathcal{R}$

Theorem (3.2): If $x_0(t)$ is a particular solution of equation (3.3) and $x_1(t), x_2(t), \dots\dots\dots x_n(t)$ is a sequence of n linearly independent solutions of equations of homogeneous ODE(3.3). Then the general solution of the equation (3.3) is given by

$X(t) = x_0(t) + c_1x_1(t) + c_2x_2(t) + \cdots c_nx_n(t)$ (3.4)

Proof: If $x(t)$ is also a solution of equation(3.3), then the function $y(t) = x(t) - x_0(t)$ solves $y'(t) = A(t)\,y$

Hence by the above theorem

$y = c_1x_1(t) + c_2x_2(t) + \cdots c_nx_n(t)$ (3.5)

Then X(t) satisfies (2-2). Conversely, for all $c_1, c_2 \dots\dots\dots\dots\dots\dots c_n$, the function (2-3) solves $y'(t) = A(t)\,y$ whence its follow that the function $X(t) = x_0(t) + y(t)$ solves (3.3).

Theorem(3.3): The general solution to the system

$x' = A(t)x + B(t)$ **(i)**

is given by $x = X(t)x \int X^{-1}(t)\,B(t)dt$ (ii)

Where X is the fundamental matrix of the system $x' = A(t)x$

Such that $\int X^{-1}(t)\,B(t)$ is a time dependent n −dimensional vector, which can be integrated in t component wise

Proof: let the matrix X satisfies the following ODE $X' = AX$,

By differentiating the equation (ii)

$$x'(t) = X'(t)\int X^{-1}B(t)dt + X(t)(X^{-1}B(t))$$
$$= AX\int X^{-1}B(t)dt + B(t)$$
$$= Ax + B(t)$$

Hence $x(t)$ solves equation (i)

let us show that the equation (ii) gives all the solutions, we have observe the integral in equation (ii) is indefinite so that it can be presented in the form

$$\int X^{-1}B(t)dt = V(t) + C$$

Where V(t) is a vector function and $C = c_1, c_2 \dots\dots\dots c_n$ is an arbitrary constant vectors

Therefore equation (i) gives

$$x(t) = X(t)V(t) + X(t)C = x_0(t) + c_1\,x_1\,(t) + c_2\,x_2\,(t) + \cdots c_n\,x_n\,(t)$$

Where $x_0(t) = X(t)V(t)$ is a solution of equation (ii)

Hence $x(t)$ is the general solution of the system

Proposition(3.1):

The set S of solutions to system $y' = Ay$, such that $A = [a_{ij}]_{n\times n} \in M_n$: with the initial condition $y(t_0) = y^0 \in R^n$ is a vector space.

Proof: Let y_1 and y_2 belong to S, the set of solutions to (i). This set is nonempty since $0\epsilon S$. For any constants $\alpha_1, \alpha_2 \epsilon R$ considering $y = \alpha_1 y_1 + \alpha_2 y_2$ We need to show that $y \in S$. This follows from the fact that $(\alpha_1 y_1 + \alpha_2 y_2)' = A(\alpha_1 y_1 + \alpha_2 y_2) \Leftrightarrow \alpha_1 y'_1 + \alpha_2 y'_2 = A\alpha_1 y_1 + A\alpha_2 y_2$ from vector space axioms

Proposition (3.2).

If $\lambda_1, \lambda_2, \dots\dots\dots\dots\dots \lambda_n$ are distinct eigenvalues of $A\epsilon M_n$ with the corresponding eigenvectors $v_1, v_2, \dots\dots\dots\dots v_m$ then $\{v_1, v_2, \dots\dots\dots\dots v_m\}$ is is linearly independent set.

Proof. Let me prove this proposition by induction. Check this for $m = 1$. v_1 is linearly independent because it is by definition non-zero. Now assume it is true for m = j, i.e., we assume that any set of j eigenvectors corresponding to distinct eigenvalues is linearly independent. Now I would like to prove that it is also true for j + 1 eigenvectors. Consider the linear combination

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_j v_j + \alpha_{j+1} v_{j+1} = 0$$

Multiply both sides of it by A from the left and use the fact that v_1 are eigenvectors:

$$\alpha_1\lambda_1 v_1 + \alpha_2\lambda_2 v_2 + \cdots \alpha_j\lambda_j v_j + \alpha_{j+1}\lambda_{j+1} v_{j+1} = 0$$

Now multiply the first equality by _1 and subtract from the second one:

$$\alpha_2(\lambda_2-\lambda_1)v_2+\cdots+\alpha_{j+1}(\lambda_{j+1}-\lambda_1)v_{j+1}=0$$

Which a linear combination of j eigenvectors, which form linearly independent set by assumptions. This means that $\alpha_i(\lambda_i-\lambda_1)=0,\, i=\,2,\ldots\ldots\ldots\ldots. j+1$
and since all λ are distinct
$\alpha_i=0,\; i=\,2,\ldots\ldots\ldots\ldots. j+1$
This leaves us with

$$\alpha_1 v_1=0\Rightarrow\alpha_1=0$$

Thus $\{v_1,v_2,\ldots\ldots\ldots. v_m\}$, is independent
Theorem (3.4): Given a system $x^{'}=Ax$ where A is a real matrix. If $x=x_1+ix_2$ is a complex solution, then its real and imaginary parts $x_1,x_2$are also solutions to the system.
Proof: since x_1+ix_2 is a solution then we have$(x_1+ix_2)^{'}$=$A(x_1+ix_2)$
By equating real and imaginary parts of this equation, $x_1^{'}=Ax_1$ and $x_2^{'}=Ax_2$
Therefore this shows that x_1, and x_2 are solutions of the system $x^{'}=Ax$
Theorem (3.5): Vector space S of the solutions to system (i) is n-dimensional.
Proof: We will prove this theorem by presenting a basis for S that has exactly n elements.
Consider n initial value problems for (i), wherey(t0) = ei; $i\,=\,1;\ldots\ldots\,;\,n$;
ande_i are standard unit vectors in R^n (i.e., vectors with one at the $i-th$ position and zeros everywhere else). Due to Theorem 1, we have n unique solutions, which I denote as y_i(t); $i\,=\,1;\ldots\ldots\ldots;\,n$
Firstly let $\{(y_1(t),y_2(t),\ldots\ldots y_n(t)\}$ span S indeed, assume that $x(t)\,\epsilon\,S$ is a solution of the system (i) along with the initial condition $x(t_0)=x^0$. Consider also $y(t)=x_1^0y_1(t)+\cdots x_n^0y_n(t)$
Therefore $y(t)$ is a solution of (i) as a linear combination of solutions, and at the point t_0y(t0) $=\,x(t$0),
Hence,by Theorem1 ,x$(t)\,\equiv\,$y(t)
This mean that any solution to (i) , (ii) can be represent as a linear combination of $\{y_1(t),\ldots\ldots\ldots\ldots..y_n(t)\}$.
Secondly, let $\{y_1(t),\ldots\ldots\ldots\ldots..y_n(t)\}$is a linearly independent set. Let $\beta_1,\beta_2\ldots\ldots\ldots..\beta_n\in R$ such that

$$\beta_1y_1(t)+\beta_2y_2(t)\ldots\ldots\ldots\ldots\ldots\beta_ny_n(t)$$

For any t (this is the definition of linear independence). Rewrite the last equality as a system in the matrix form:
$\Phi(t)\beta=0$
Where Φ(t) is the matrix having $y_i(t)$ as its $i-th$ column, and $\beta^T=(\beta_1,\beta_2,\ldots\ldots\ldots\ldots\beta_n)$
Since the last equality has to be true for any t then it is true for t_0, but for $t_0=t$ and $\Phi(t_0)=I$and the only solution to$I\beta=0$,
is a trivial one, $\beta^T=(0,0,\ldots\ldots\ldots\ldots 0)$, Therefore, $\{y_1(t),\ldots\ldots\ldots\ldots..y_n(t)\}$is a linearly independent set. Since$\{y_1(t),\ldots\ldots\ldots\ldots..y_n(t)\}$ span S and is linearly independent, then it is a basis for S.
3.2.1 The Implementation of The Method In Solving Inhomogeneous Linear ODEs Systems.
(i) First order linear differential systems.
Consider the first order of linear differential equations

$$x_1^{'}=-x_2\,,x_2^{'}=x_1$$

Find the solution of the systems above.
Solution:
We can reform the linear system as $x^{'}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}x$ so the system has two independent solutions, such that
$x_1(t)=\begin{pmatrix}cost\\sint\end{pmatrix}andx_2(t)=\begin{pmatrix}-sint\\cost\end{pmatrix}$
Then the corresponding fundamental matrix is
$\mathrm{X}=\begin{pmatrix}\mathrm{cost}&-\mathrm{sint}\\\mathrm{sint}&\mathrm{cost}\end{pmatrix}andX^{-1}=\begin{pmatrix}\mathrm{cost}&\mathrm{sint}\\-\mathrm{sint}&\mathrm{cost}\end{pmatrix}$
Consider the ODE $x^{'}=A(t)x+B(t)$, where $B(t)=\begin{pmatrix}b_1(t)\\b_2(t)\end{pmatrix}$, so by equation (1) we get the general solution as

$$x(t)=\begin{pmatrix}\mathrm{cost}&-\mathrm{sint}\\\mathrm{sint}&\mathrm{cost}\end{pmatrix}\int\begin{pmatrix}\mathrm{cost}&\mathrm{sint}\\-\mathrm{sint}&\mathrm{cost}\end{pmatrix}\begin{pmatrix}b_1(t)\\b_2(t)\end{pmatrix}dt=\begin{pmatrix}\mathrm{cost}&-\mathrm{sint}\\\mathrm{sint}&\mathrm{cost}\end{pmatrix}\int\begin{pmatrix}b_1(t)cost+b_2(t)sint\\-b_1(t)sint+b_2(t)cost\end{pmatrix}dt$$

Let a particular example $(t)=\begin{pmatrix}1\\-1\end{pmatrix}$, then the integral is

$$\int\begin{pmatrix}cost&-tsint\\-sint&-tcost\end{pmatrix}dt=\begin{pmatrix}tcost+C_1\\-tsint+C_2\end{pmatrix}$$

Whence $x(t) = \begin{pmatrix} \text{cost} & -\text{sint} \\ \text{sint} & \text{cost} \end{pmatrix} \begin{pmatrix} tcost + C_1 \\ -tsint + C_2 \end{pmatrix} = \begin{pmatrix} C_1\text{cost} - C_2\text{sint} \\ C_1 sint + C_2 cost \end{pmatrix}$

$$x(t) = \begin{pmatrix} t \\ 0 \end{pmatrix} + C_1 \begin{pmatrix} cost \\ sint \end{pmatrix} + C_2 \begin{pmatrix} -sint \\ cost \end{pmatrix}$$

A scalar ODE of n-th order

3.3 Solution of Inhomogeneous Linear Differential Equations Systems of Order n

$x^{(n)} + a_1(t)x^{(n-1)} + \cdots a_n(t)x = f(t)$(3.6)

Where $a_k(t)$ and $f(t)$ are continuous function on some interval I. it can be reduce to the vector ODE as

$$x' = A(t)x + B(t)$$

$$X(t) = \begin{pmatrix} x(t) \\ x'(t) \\ . \\ .x^{(n-1)}(t) \end{pmatrix}, A = \begin{pmatrix} 0 \ 1 \ 0 \ 0 \ldots\ldots\ldots 0 \\ 0 \ 0 \ 1 \ldots\ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 0 \ \ 0 \ \ 0 \ \ldots\ldots\ldots 1 \\ -a_n \ -a_{n-1} - a_1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ f \end{pmatrix} \quad (3.7)$$

If x_1.............x_2 are$n$ linearly independent solutions to the homogeneous ODE

$$x^{(n)} + a_1(t)x^{(n-1)} + \cdots a_n(t)x = 0$$

Then denoting by x_1.............$x_2$the corresponding vector solution, we obtain the fundamentalmatrix

$$X = (x_1|x_2| \ldots\ldots..|x_n) = \begin{pmatrix} x_1 x_2 \ldots\ldots..x_n \\ x_1' \ x_2' \ \ldots\ldots\ldots x_n' \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_1^{(n-1)} x_2^{(n-1)} x_n^{(n-1)} \end{pmatrix} \quad (3.8)$$

By multiplying $X^{-1}$by B, where y_{ik} the element of X^{-1}at position i, k so that i is the row index and k is the column index Denote also by y_k the $k-th$ columnof X^{-1}, that is,

$$y_k = \begin{pmatrix} y_{1k} \\ . \\ . \\ y_{nk} \end{pmatrix}, \text{ then } X^{-1}B = \begin{pmatrix} y_{11}y_{12} \ldots\ldots. y_{1n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_{n1}y_{n2} \ldots\ldots\ldots y_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ . \\ . \\ f \end{pmatrix} = \begin{pmatrix} y_{1n}f \\ . \\ . \\ y_{nn}f \end{pmatrix} = f y_n \quad (3.9)$$

and the general vector solution is

$x = X(t) \int f(t) y_n(t) dt$ (3.10)

To find the function x(t)which is the first component of x by taking the first row of X to multiply by the columnvector$\int f(t)y_n(t)dt$, where

$$x(t) = \sum_{j=0}^{\infty} x_j(t) \int f(t) y_{jn}(t) dt \quad (3.11)$$

Corollary: Let $x_1 \ldots\ldots..x_n$ be n linearly independent solution to $x^{(n)} + a_1(t)x^{(n-1)} + \cdots a_n(t)x = 0$ and X be the corresponding fundamental matrix.Then, for any continuous function $f(t)$,the general solution to the ODE

$x^{(n)} + a_1(t)x^{(n-1)} + \cdots a_n(t)x = f(t)$

is given by

$$x(t) = \sum_{j=0}^{\infty} x_j(t) \int f(t) y_{jn}(t) dt \quad (3.12)$$

Wherey_{jk} are the entries of the matrix X^{-1}.

(I) IMPEMENTATION OF THE METHOD

Solve the inhomogeneous linear differential equation below

$x'' = \sin(t) - x$ (i)

Solution:

Rewrite the equation (i) as equation (3.6)

$x'' + x = \sin(t)$ (ii)

The independent solutions for homogeneous linear differential equation are

Hence $x_1(t) = cost$, $x_2(t) = sint$

Where $X = \begin{pmatrix} cost & sint \\ -sint & cost \end{pmatrix}$ and $X^{-1} = \begin{pmatrix} cost & -sint \\ sint & cost \end{pmatrix}$

So the particular solution for equation(ii) using the equation (3.12)

$$x(t) = \sum_{j=0}^{\infty} x_j(t) \int f(t) y_{jn}(t) dt = x_1(t) \int f(t) y_{12}(t) dt + x_2(t) \int f(t) y_{22}(t) dt$$

$$x\,(t) = cost\int sint(-sint)dt + sint\int sint(cost)dt$$
$$= -cost\int sin^2tdt + \frac{1}{2}sint\int sint(2t)dt$$
$$= -cost\left(\frac{1}{2}t - \frac{1}{4}sin2t + c_1\right) + \frac{1}{4}sint(-cos2t + c_2)$$
$$= -\frac{1}{2}tcost + \frac{1}{4}(sin2t + cost - sintcos2t) + c_3cost + c_4sint$$
$$= -\frac{1}{2}tcost + c_3cost + c_5sint$$

Therefore $x\,(t)$ is the solution of system of linear differential equations

4. Discussion:

The study introduced the solution of linear differential equations systems in homogeneous and inhomogeneous form using coefficient matrices techniques which by generating eigevalues, eigenvectors of the matrices. Its powerful method,and gives typical results comparing with other analytical techniques. The major drawback of this method, is extracting the inverse of coefficients matrices, the process is so tedious and sophisticate to generate for higher order matrices.

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