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Some geometry properties of a class of Analytic function with Gegenbauer polynomial

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Abstract

A class of analytic functions involving the Gegenbauer polynomial in the unit disk was introduced and investigated . Coefficient bounds fot the function of the class $S(\alpha, x)(z)$ was obtained , furthermore , upper bounds of the second and third Toeplitz determinant belongs to the class $S(\alpha, x)(z)$ were established. Various known and new result are also derived..

Keywords: Toeplitz determinant, Analytic functions, Gegenbauer Polynomials, coefficient bounds, subordination.

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1 Introduction and Preliminaries

Let A denote the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1.1}$$

which are analytic in the open disk Δ={z $\in \mathbb{C}$: $|z| < 1$}, and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Furthermore, let S be the class of all functions in A which are *univalent* in Δ

A *subordination* between two analytic functions f and g is written as f $\prec$ g. conceptually the analytic function f is subordinate to g if the image under g contain the image under f. Technically, the analytic function f is subordinate to g if there exists a schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all z $\in \Delta$; such that

$$f(z) = g(\omega(z)).$$

Besides, if the function g is univalent in Δ, then the following equivalence holds:

$$f(z) \prec g(z) \text{ if and only if } f(0) = g(0) \text{ and } f(\Delta) \subset \text{g}(\Delta).$$

Further on subordination principle see details in [1]

A Toeplitz determinants is an upside down Hankel determinants, that is Hankel determinants have constant entries along the reverse diagonal while Toeplitz determinants have constant entries the diagonal.

Thomas and Halim [6] introduced the symmetric Toeplitz determinant $T_q(n)$ for analytic function $f(z)$ of the form $f(z) = z + a_2z^2 + a_3z^3 + ...$ defined as follows;

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_n & \dots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \dots & a_n \end{vmatrix}$$

(where $n, q = 1, 2, 3, ...a_1 = 1$ for f(z) $\in$ S)

In particular,

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, T_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & a_2 \\ a_3 & a_2 & a_1 \end{vmatrix}$$

We assume $a_1 = 1$, we have

$T_3(1) = a_3(a_2^2 - a_3) - a_2(a_2 - a_2a_3) + (1 - a_2^2)$.

Then,

$$T_3(1) \leq |a_3||a_2^2 - a_3| + |a_2||a_2 - a_2a_3| + |1 - a_2^2|$$

see details in [6, 11]

For non-zero real constant α, a generating function of Gegenbauer polynomials is defined by

$$\kappa_\alpha(x, z) = \frac{1}{(1 - 2xz + z^2)^\alpha}, \tag{1.2}$$

where x $\in$ [-1,1] and z $\in \Delta$. For fixed x the function κ_α is analytic in Δ, so it can be expanded on a Taylor series as

$$\kappa_\alpha(x, z) = \sum_{n=0}^{\infty} C_n^\alpha(x)z^n, \tag{1.3}$$

Where $C_n^\alpha(x)$ is Gegenbauer polynomial of degree n.

Obviously κ_α generates nothing when $\alpha = 0$. Therefore, the generating function of the Gegenbauer polynomial is set to be

$$\kappa_0(x, z) = 1 - log(1 - 2xz + z^2) = \sum_{n=0}^{\infty} C_n^0(x)z^n \tag{1.4}$$

for $\alpha = 0$ and Gegenbauer polynomials can also be defined by the following recurrence relations:

$$C_n^\alpha(x) = \frac{1}{n}[2x(n + \alpha - 1)C_{n-1}^\alpha(x) - (n + 2a - 2)C_{n-1}^\alpha(x)], \tag{1.5}$$

with initial values

$C_0^\alpha(x) = 1$, $C_1^\alpha(x) = 2\alpha x$ and $C_2^\alpha(x) = 2\alpha(1 + \alpha)x^2 - \alpha$. see details in[7]

Definition 1 *A function f(z) $\in$ A given by (1.1) is in the class $S(\alpha, x)(z)$ if*

$$f'(z) + \frac{1 + e^{it}}{2} zf''(z) \prec \kappa_\alpha(x, z)$$

where $0 \leq \alpha < 1$, $x \in (\frac{1}{2}, 1]$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ *and* $z \in \Delta$

The Lemmas below are employed for the purpose of this research

Lemma 1 *if the function* $P(z) = 1 + \sum_{n=1}^{\infty} C_n p_n \in P$, *then*
$|c_n| \leq 2$, $n \geq 1$
The inequality is sharp for $f(z) = \frac{1+z}{1-z}$

Lemma 2 *The power series for* $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. *Let the function* $p \in \Delta$ *be given by (1.1), then,*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{1.6}$$

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)\eta \tag{1.7}$$

for some value of x,$\eta \in \mathbb{C}$ with $|x| \leq 1$ and $|\eta| \leq 1$. see details in [7, 12]

Recently, Ala Amourah *et al.* [7] examined the Gegenbauer polynomials (or Ultraspherical polynomials) $C_\iota^\alpha(x)$. They are orthogonal polynomials on [-1,1] that can be defined by the recurrence relation

$$C_\iota^\alpha(x) = \frac{2x(\iota + \alpha - 1)C_{\iota-1}^\alpha(x) - (\iota + 2\alpha - 2)C_{\iota-2}^\alpha(x)}{\iota}, C_0^\alpha(x) = 1, C_1^\alpha(x) = 2\alpha x \tag{1.8}$$

where $\iota \in \mathbb{N}\backslash\{1\}$.it si easy to see from (2.8) that $C_2^\alpha(x) = 2\alpha(1+\alpha)x^2 - \alpha$. For $\alpha \in \mathbb{R}\backslash\{0\}$, a generating function of the sequence $C_\iota^\alpha(x)$, j $\in \mathbb{N}$, is defined by

$$\kappa_\alpha(x, z) := \sum_{\iota=0}^{\infty} C_\iota^\alpha(x) z^\iota = \frac{1}{(1 - 2xz + z^2)^\alpha}, \tag{1.9}$$

where z $\in \Delta$ and x $\in$ [-1,1]. See[3] for details.

Two particular case of $C_\iota^\alpha(x)$ are

i) $C_\iota^1(x)$ the second kind of chebyshev polynomials and

ii) $C_\iota^{\frac{1}{2}}(x)$ the Legendre polynomials (see details in [4,7])

2 Coefficient Bounds

In this work, the coefficient bounds and upper bounds of second and third Toeplitz determinant for the function in class $S(\alpha, x)(z)$,the following theorems are investigated:

Theorem 2.1 *If the function f(z)* $\in S(\alpha, x)(z)$. *Then*

$$a_2 = \frac{c_1^\alpha c_1}{2\phi_2}$$

$$a_3 = \frac{c_1^\alpha(x)c_2 + c_2^\alpha(x))c_1^2}{3\phi_3}$$

$$a_4 = \frac{c_1^\alpha(x)c_3 + c_3^\alpha(x)c_1^3}{4\phi_4}$$

Where $\phi_j = \left[1 + \frac{1+e^{it}}{2}(j-1)\right]$ *and* $j \in \mathbb{N}\{2\}$

Proof: For f(z) $\in S_q(\alpha, x)(z)$ then there exist $\omega(z)$ called a schwarz function with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$f'(z) + \frac{1+e^{it}}{2} z f''(z) = \kappa_\alpha(x, \omega(z)) \tag{2.1}$$

Now, insert the value of $f'(z)$ and $f''(z)$ in (2.1) we have

$$1 + \sum_{j=2}^{\infty} j a_j z^{j-1} + \sum_{j=2}^{\infty} j(j-1) a_j \frac{1+e^{it}}{2} z^{j-1}$$

$$1 + \sum_{j=2}^{\infty} j a_j z^{j-1} \left[1 + \frac{1+e^{it}}{2}(j-1) \right] \tag{2.2}$$

(2.2) becomes

$$1 + \sum_{j=2}^{\infty} j \phi_j a_j z^{j-1} \tag{2.3}$$

where $\phi_j = \left[1 + \frac{1+e^{it}}{2}(j-1) \right]$ and j $\in \mathbb{N}\{2\}$
For some analytic functions
$\omega(z) = c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + ...$
On the unit disk Δ with $\omega(0) = 0$,$|\omega(z)| < 1$ (z $\in \Delta$) then,

$$\kappa_\alpha(x, \omega(z)) = 1 + c_1^\alpha(x) c_1 z + [c_1^\alpha(x) c_2 + c_2^\alpha(x) c_1^2] z^2 + [c_1^\alpha(x) c_3 + c_3^\alpha(x) c_1^3] z^3 + ... \tag{2.4}$$

Equating (2.3) and (2.4) we have

$$1 + \frac{z f''(z)}{f'(z)} = \kappa_\alpha(\omega(z))$$

$1 + 2\phi_2 a_2 z + 3\phi_3 a_3 z^2 + 4\phi_4 a_4 z^3 + ... = 1 + c_1^\alpha(x) c_1 z + [c_1^\alpha(x) c_2 + c_2^\alpha(x) c_1^2] z^2 + [c_1^\alpha(x) c_3 + c_3^\alpha(x) c_1^3] z^3 + ...$
Equating the coefficient of z, z^2 and z^3 we get

$$a_2 = \frac{c_1^\alpha c_1}{2\phi_2}, \tag{2.5}$$

$$a_3 = \frac{c_1^\alpha(x) c_2 + c_2^\alpha(x)) c_1^2}{3\phi_3}, \tag{2.6}$$

$$a_4 = \frac{c_1^\alpha(x) c_3 + c_3^\alpha(x) c_1^3}{4\phi_4} \tag{2.7}$$

■

Theorem 2.2 *Let $0 \leq \alpha < 1$, and if the function f(z) be of the form (1.1) belongs to the class $S(\alpha, x)(z)$*

$$|a_3^2 - a_2^2| \leq \left| C_1^{2\alpha} \left(\frac{9}{4\phi_3^3} - \frac{1}{\phi_2^2} \right) + \frac{16 C_2^{2\alpha}(x)}{9\phi_3^2} \right|$$

Proof: Using the coefficients in the Theorem 2.1 then $T_2(2) \leq |a_3^2 - a_2^2|$ becomes

$$\left|a_3^2 - a_2^2\right| = \left|\frac{C_1^{2\alpha}(x)C_2^2}{9\phi_3^2} - \frac{C_1^2\left(9\phi_3^2 C_1^{2\alpha}(x) - 4\phi_2^2 C_2^{2\alpha}(x)\right)}{36\phi_3^2\phi_2^2}\right| \tag{2.8}$$

using lemma 2 in (2.8) we have

$$\left|a_3^2 - a_2^2\right| = \left|\frac{C_1^{2\alpha}(x)}{9\phi_3^2}\left(\frac{C_1^4 + 2C1^2xX + x^2X^2}{4}\right) - \frac{C_1^2\left(9\phi_3^2 C_1^{2\alpha}(x) - 4\phi_2^2 C_2^{2\alpha}(x)\right)}{36\phi_3^2\phi_2^2}\right| \tag{2.9}$$

simplifying (2.9),we get

$$\left|a_3^2 - a_2^2\right| \leq \left|\frac{C_1^{2\alpha}(x)C_1^4}{36\phi_3^2} - \frac{C_1^{2\alpha}(x)C_1^2}{4\phi_2^2} + \frac{C_2^{2\alpha}(x)}{C}{}_1^4 9\phi_3^2\right| + \frac{C_1^{2\alpha}(x)C_1^2|x|X}{18\phi_3^2} + \frac{C_1^{2\alpha}(x)|x|^2X^2}{36\phi_3^2} = \upsilon(|x|, c) \tag{2.10}$$

Differentiating (2.10) partially with respect to $|x|$ we get

$$\left|a_3^2 - a_2^2\right| \leq \left|\frac{C_1^{2\alpha}(x)C_1^4}{36\phi_3^2} - \frac{C_1^{2\alpha}(x)C_1^2}{4\phi_2^2} + \frac{C_2^{2\alpha}(x)}{C}{}_1^4 9\phi_3^2\right| + \frac{C_1^{2\alpha}(x)C_1^2X}{18\phi_3^2} + \frac{C_1^{2\alpha}(x)|x|X^2}{18\phi_3^2} \tag{2.11}$$

suppose $|x| = 1$ then

$$\left|a_3^2 - a_2^2\right| \leq \left|\frac{C_1^{2\alpha}(x)C^4}{36\phi_3^2} - \frac{C_1^{2\alpha}(x)C^2}{4\phi_2^2} + \frac{C_2^{2\alpha}(x)}{C}^4 9\phi_3^2\right| + \frac{C_1^{2\alpha}(x)}{18\phi_3^2}(C^2X + X^2)$$

since $X = (4 - C_1^2)$ and $c \in [0, 2]$ then

$$\left|a_3^2 - a_2^2\right| \leq \left|\frac{C_1^{2\alpha}(x)C^4}{36\phi_3^2} - \frac{C_1^{2\alpha}(x)C^2}{4\phi_2^2} + \frac{C_2^{2\alpha}(x)}{C}^4 9\phi_3^2\right| + \frac{C_1^{2\alpha}(x)}{18\phi_3^2}(16 + 4C^2)$$

Obtaining the maximum value for $|a_3^2 - a_2^2|$ at $c[0, 2]$ i.e $c \leq 2$ we have

$$\left|a_3^2 - a_2^2\right| \leq \left|C_1^{2\alpha}\left(\frac{9}{4\phi_3^3} - \frac{1}{\phi_2^2}\right) + \frac{16C_2^{2\alpha}(x)}{9\phi_3^2}\right|$$

■

Theorem 2.3 *Let* $0 \leq \alpha < 1$, *and if the function f(z) be of the form (1.1) belongs to the class then* $S(\alpha, x)(z)$

$$T_3(1) \leq \left|1 + \frac{4C_1^{3\alpha}(x) + 8C_1^{2\alpha}(x)C_2^{\alpha}(x)}{3\phi_2^2\phi_3^2} - \frac{2C_1^{2\alpha}(x)}{\phi_2^2} - \left(\frac{4C_1^{2\alpha}(x) + 16C_2^{2\alpha}(x)}{9\phi_3^2}\right)\right|$$

Proof: Using the coefficients in the Theorem (2.1) then
$T_3(1) = \left|1 + 2a_2^2(a_3 - 1) - a_3^2\right|$ becomes

$$T_3(1) = \left|1 + \frac{C_1^{3\alpha}(x)C_1^2C_2 + C_1^{2\alpha}(x)C_2^{\alpha}(x)C_1^4 - 3C_1^{2\alpha}(x)C_1^2\phi_2}{6\phi_2^2\phi(3)} - \frac{C_1^{2\alpha}(x)C_2^2 + C_2^{2\alpha}(x)C_1^4}{9\phi_3^2}\right| \tag{2.12}$$

using lemma 2 in (2.10) we have

$$T_3(1) = \left| 1 + \frac{C_1^{3\alpha}(x)C_1^4}{12\phi_2^2\phi_3} + \frac{C_1^{3\alpha}(x)C_1^2x}{3\phi_2^2\phi_3} - \frac{C_1^{3\alpha}(x)C_1^4x}{12\phi_2^2\phi_3} + \frac{C_1^{2\alpha}(x)C_2^{\alpha}(x)C_1^4}{6\phi_2^2\phi 3} - \frac{C_1 2\alpha(x)C_1^2}{2\phi_2^2} - \frac{C_1^{2\alpha}(x)C_1^4}{36\phi_3^2} - \frac{C_1^{2\alpha}(x)C_1^2xX}{18\phi_3^2} - \frac{C_1^{2\alpha}(x)x^2X^2}{36\phi_3^2} - \frac{C_1^{2\alpha}(x)C_4}{9\phi_3^2} \right|$$

We assume that $X = (4 - C_1^2)$

Applying Triangular Inequality and let $c_1 = c$,

$$T_3(1) \le \left| 1 + \frac{C1^{3\alpha}(x)C^4}{12\phi_2^2\phi_3} + \frac{C_1^{2\alpha}(x)C_2^{\alpha}(x)C^4}{6\phi_2^2\phi 3} - \frac{C_1^{2\alpha}(x)C^2}{2\phi_2^2} - \frac{C_1^{2\alpha}(x)C^4}{36\phi_3^2} - \frac{C_1^{2\alpha}(x)C^4}{9\phi_3^2} \right| + \frac{C_1^{3\alpha}(x)C_1^2|x|}{3\phi_2^2\phi_3} - \frac{C_1^{3\alpha}(x)C_1^4|x|}{12\phi_2^2\phi_3} - \frac{2C_1^{2\alpha}(x)C^2|x|}{9\phi_3^2} + \frac{C_1^{2\alpha}(x)C^4|x|}{18\phi_3^2} - \frac{4C_1^{2\alpha}(x)|x|^2}{9\phi_3^2} + \frac{2C_1^{2\alpha}(x)C^2|x|^2}{9\phi_3^2} - \frac{C_1^{2\alpha}(x)C^4|x|^2}{36\phi_3^2} = \upsilon(c, |x|)$$

Differentiating $\upsilon(c, |x|)$ partially with respect to $|x|$ and clearly $\varphi'(|x|) > 0$ on $[0, 1]$ which implies $\varphi(|x|, c) \le \varphi(1, c)$w

$$T_3(1) \le \left| 1 + \frac{C_1^{3\alpha}(x)C^4}{12\phi_2^2\phi_3} + \frac{C_1^{2\alpha}(x)C_2^{\alpha}(x)C^4}{6\phi_2^2\phi 3} - \frac{C_1 2\alpha(x)C^2}{2\phi_2^2} - \frac{C_1^{2\alpha}(x)C^4}{36\phi_3^2} - \frac{C_1^{2\alpha}(x)C^4}{9\phi_3^2} \right| + \frac{C_1^{3\alpha}(x)C_1^2}{3\phi_2^2\phi_3} - \frac{C_1^{3\alpha}(x)C_1^4}{12\phi_2^2\phi_3} - \frac{2C_1^{2\alpha}(x)C^2}{9\phi_3^2} + \frac{C_1^{2\alpha}(x)C^4}{18\phi_3^2} - \frac{8C_1^{2\alpha}(x)|x|}{9\phi_3^2} + \frac{4C_1^{2\alpha}(x)C^2|x|}{9\phi_3^2} - \frac{C_1^{2\alpha}(x)C^4|x|}{18\phi_3^2} \quad (2.13)$$

since $|x| = 1$ and $C \in [0, 2]$ then (2.11) becomes

$$T_3(1) \le \left| 1 + \frac{4C_1^{3\alpha}(x) + 8C_1^{2\alpha}(x)C_2^{\alpha}(x)}{3\phi_2^2\phi_3^2} - \frac{2C_1^{2\alpha}(x)}{\phi_2^2} - \left(\frac{4C_1^{2\alpha}(x) + 16C_2^{2\alpha}(x)}{9\phi_3^2} \right) \right| + \frac{C_1^{3\alpha}(x)C_1^2}{3\phi_2^2\phi_3} - \frac{C_1^{3\alpha}(x)C_1^4}{12\phi_2^2\phi_3} - \frac{2C_1^{2\alpha}(x)C^2}{9\phi_3^2} + \frac{C_1^{2\alpha}(x)C^4}{18\phi_3^2} - \frac{8C_1^{2\alpha}(x)}{9\phi_3^2} + \frac{4C_1^{2\alpha}(x)C^2}{9\phi_3^2} - \frac{C_1^{2\alpha}(x)C^4}{18\phi_3^2} \quad (2.14)$$

Obtaining the maximum value for $|T_3(1)|$ at $c[0, 2]$ i.e $c \le 2$ we have

$$T_3(1) \le \left| 1 + \frac{4C_1^{3\alpha}(x) + 8C_1^{2\alpha}(x)C_2^{\alpha}(x)}{3\phi_2^2\phi_3^2} - \frac{2C_1^{2\alpha}(x)}{\phi_2^2} - \left(\frac{4C_1^{2\alpha}(x) + 16C_2^{2\alpha}(x)}{9\phi_3^2} \right) \right| \quad (2.15)$$

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