Some properties of a matrix of certain magic square

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The purpose of this article is to discover some poperties of a matrix of magic square, we restrict our research in three by three matrix of magic square, they have astonishing porperties such as their determinant and their Eign values.

Introduction

The following three by three magic square is constructed in [1],[2] of constant sum c and sequenc-difference d as follows

$c/3-3d$	c/3+2d	$c/3+d$
$c/3+4d$	c/3	$c/3-4d$
$c/3-d$	c/3 -2d	$c/3+3d$

Consider a matrix of the above magic square with constant sum c and sequence-difference d

$$A=\begin{bmatrix} \frac{c}{3}-3d & \frac{c}{3}+2d & \frac{c}{3}+d \\ \frac{c}{3}+4d & \frac{c}{3} & \frac{c}{3}-4d \\ \frac{c}{3}-d & \frac{c}{3}-2d & \frac{c}{3}+3d \end{bmatrix}$$

The main results

The matrix A^2 does not form a magic square, because its elements are repeated as follows:

$$A^2=\begin{bmatrix} \frac{c^2}{3}+16d^2 & \frac{c^2}{3}-8d^2 & \frac{c^2}{3}-8d^2 \\ \frac{c^2}{3}-8d^2 & \frac{c^2}{3}+16d^2 & \frac{c^2}{3}-8d^2 \\ \frac{c^2}{3}-8d^2 & \frac{c^2}{3}-8d^2 & \frac{c^2}{3}+16d^2 \end{bmatrix}$$

We can deduce that $A^{2n}, n=1,2,3,\ldots$ is are not magic squares, but

$$A^3=\begin{bmatrix} \frac{c^3}{3}-72d^3 & \frac{c^3}{3}+48d^3 & \frac{c^3}{3}+24d^3 \\ \frac{c^3}{3}+96d^3 & \frac{c^3}{3} & \frac{c^3}{3}-96d^3 \\ \frac{c^3}{3}-24d^3 & \frac{c^3}{3}-48d^3 & \frac{c^3}{3}+72 \end{bmatrix}$$

which is a magic square of constant sum c^3.

If A is a magic square of constant sum c and sequence-difference d_1 , and B is another matrix of magic square of constant sum k and sequence-difference d_2, then they are commutativee that is if

$$A = \begin{bmatrix} \frac{c}{3} - 3d_1 & \frac{c}{3} + 2d_1 & \frac{c}{3} + d_1 \\ \frac{c}{3} + 4d_1 & \frac{c}{3} & \frac{c}{3} - 4d_1 \\ \frac{c}{3} - d_1 & \frac{c}{3} - 2d_1 & \frac{c}{3} + 3d_1 \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{k}{3} - 3d_2 & \frac{k}{3} + 2d_2 & \frac{k}{3} + d_2 \\ \frac{k}{3} + 4d_2 & \frac{k}{3} & \frac{k}{3} - 4d_2 \\ \frac{k}{3} - d_2 & \frac{k}{3} - 2d_2 & \frac{k}{3} + 3d_2 \end{bmatrix}$$

then

$$AB = BA = \begin{bmatrix} \frac{ck}{3} + 16d_1d_2 & \frac{ck}{3} - 8d_1d_2 & \frac{ck}{3} - 8d_1d_2 \\ \frac{ck}{3} - 8d_1d_2 & \frac{ck}{3} + 16d_1d_2 & \frac{ck}{3} - 8d_1d_2 \\ \frac{ck}{3} - 8d_1d_2 & \frac{ck}{3} - 8d_1d_2 & \frac{ck}{3} + 16d_1d_2 \end{bmatrix}$$

It is clear thar A and B are commutative and the elements of AB are repeated which means AB is not a magic square.

In genral the even power of the matrix A is not a magic square, but if n odd, then the matrix A^n is a magic square as follows

$$A^n = \begin{bmatrix} \frac{c^n}{3} - 3m & \frac{c^n}{3} + 2m & \frac{c^n}{3} + m \\ \frac{c^n}{3} + 4m & \frac{c^n}{3} & \frac{c^n}{3} - 4m \\ \frac{c^n}{3} - m & \frac{c^n}{3} - 2m & \frac{c^n}{3} + 3m \end{bmatrix} \quad (1)$$

where $m = (24)^{\frac{n-1}{2}} d^n$

Some examples

If

$$A = \begin{bmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{bmatrix} \quad (2)$$

wich is of constant sum 15 and sequence-difference 1, then det $A = -24 \times 15 \times 1 = -360$.

If

$$B = \begin{bmatrix} -1 & 9 & 7 \\ 13 & 5 & -3 \\ 3 & 1 & 11 \end{bmatrix} \quad (3)$$

which is of constant sum 15 and sequence-difference is equal to 2, then det$B = -24 \times 15 \times 4 = -1440$.

and

$$B^3 = \begin{bmatrix} 549 & 1509 & 1317 \\ 1893 & 1125 & 357 \\ 933 & 741 & 1701 \end{bmatrix} \quad (4)$$

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so $\det B^3 = -(24)^3 15^3 (2^6) = -2985984000$

The deteriminat of the matrix $A^n = -24^n c^n d^{2n}$ and A^n has three Eignvalues are all real and equal to $c^n, (24)^{\frac{n}{2}} d^n$ and $-(24)^{\frac{n}{2}} d^n$.

Because the charcterisc equation of matrix (1) is $(c-\lambda)(\lambda^2 - (24)^n d^{2n})$ which means all Eignvalues of the matrix A^n are real and are equal to $c, (24)^{n/2} d^n$, and $-(24)^{n/2} d^n$.

The magic square matrix (2) has A detrminant -360 and Eignvalus 15, $\sqrt{24}$ and - $\sqrt{24}$

The matrix of magic square

$$B = \begin{bmatrix} 2 & 12 & 10 \\ 16 & 8 & 0 \\ 6 & 4 & 14 \end{bmatrix}$$

in this case $d = 2$, $c = 24$ so det $B = -24 \times 24 \times 24$ =-2304 and its Eignvalues are 24, 2 $\sqrt{24}$ and $-2\sqrt{24}$

The matrix of magic square

$$B = \begin{bmatrix} -3 & 12 & 9 \\ 18 & 6 & -6 \\ 3 & 0 & 15 \end{bmatrix}$$

has constant sum $c = 18$ and sequence-difference $d = 3$ its determinant is $-24 \times 18 \times 9 = -3888$ and its Eignvalues are $18, 3\sqrt{24}$ and $-3\sqrt{24}$

It is clear that the trace of a matrix of a magic s quare is the constant s sum c

References

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