Sums of powers of consecutive triangular numbers

Several methods of computing sums of powers of consecutive triangular numbers exist. This letter presents an easy method that uses Newton's difference formula for polynomials, that is,

$$f(n) = f_0 + \binom{n}{1}\Delta f_0 + \binom{n}{2}\Delta^2 f_0 + \binom{n}{3}\Delta^3 f_0 + \ldots + 0.$$

If we want to find a formula for $1 + 3 + 6 + \ldots + T_n$, that is, to find a function $f : \{0, 1, 2, 3, 4\} \to \{0, 1, 4, 10, 20\}$ and use forward differences in this case, $f(0) = 0, \Delta f_0 = f_1 - f_0 = 1, \Delta^2 f_0 = \Delta f_1 - \Delta f_0 = 2, \Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0 = 1, \Delta^4 f_0 = 0$, so the formula for the sum of of the first triangular number is in this case the power is one $S_1 = \binom{n}{1} + 2\binom{n}{2} + \binom{n}{3} = n + n(n-1) + \frac{n(n-1)(n-2)}{6}$. After simplification,

$$S_1 = \frac{n(n+1)(n+2)}{6}.$$

To find a formula for the sum

$$1^2 + 3^2 + 6^2 + \ldots + T_n^2,$$

that is to find a function $f : 0, 1, 2, 3, 4, 5, 6 \to \{0, 1, 10, 46, 146, 371, 812\}$, in this case $f(0) = 0, \Delta f_0 = 1, \Delta^2 f_0 = 8, \Delta^3 f_0 = 19, \Delta^4 f_0 = 18, \Delta^5 f_0 = 6, \Delta^6 f_0 = 0$, so the formula is

$$S_2 = \binom{n}{1} + 8\binom{n}{2} + 19\binom{n}{3} + 18\binom{n}{4} + 6\binom{n}{5}.$$

After simplification,

$$S_2 = \frac{n(n+1)(n+2)(3n^2+6n+1)}{60}$$

To find a formula for the sum

$$1^3 + 3^3 + 6^3 + 10^3 + 15^3 + 21^3 + 28^3 + \ldots + T_n^3$$

that is to find a function

$$f : \{0, 1, 2, 3, 4, 5, 6, 7, 8\} \to \{0, 1, 28, 244, 1244, 4619, 13880, 35832, 82488\}$$

$f(0) = 0, \Delta f_0 = 1, \Delta^2 f_0 = 26, \Delta^3 f_0 = 163, \Delta^4 f_0 = 432, \Delta^5 f_0 = 564, \Delta^6 f_0 = 360, \Delta^7 f_0 = 90, \Delta^8 f_0 = 0.$

$$S_3 = \binom{n}{1} + 26\binom{n}{2} + 163\binom{n}{3} + 432\binom{n}{4} + 564\binom{n}{5} + 360\binom{n}{6} + 90\binom{n}{7}$$

After simplification

$$S_3 = \frac{n(n+1)(n+2)(15n^4 + 60n^3 + 63n^2 + 6n - 4)}{840}.$$

In general to find a formula for the sum

$$1^k + 3^k + 6^k + 10^k + 15^k + \ldots + T^k$$

just to find a function $f : \{0, 1, 2, \ldots, 2k+2\} \to \{0, 1, f(2), f(3), \ldots f(2k+2)\}$, note that $f(0) = 0$ and $f(1) = 1$, $\Delta f_0 = 1$ find $\Delta^2 f_0, \Delta^2 f_0, \ldots, \Delta^{2k+1}$ and note that $\Delta^{2k+2} f_0 = 0$. Applying Newton's formula, we have it.

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