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THE DOUBLE LAPLACE-ABOODH TRANSFORM AND THEIR PROPERTIES WITH APPLICATIONS

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ABSTRACT. In this paper, we present a new operator integral transform called double Laplace-Aboodh transform, some valuable properties for the transform are current. Furthermore, we use this transform for solving some linear partial differential equations.

1. INTRODUCTION

The source of the integral transforms can be traced back to the work of Laplace in 1780s and Fourier in 1822 [12]. Laplace transform is highly competent for solving some class of ordinary and partial differential equations. The integral and differential equations have been solved using many integral transforms. Aboodh transform was introduced by Khalid Aboodh to facilitate the process of solving ordinary and partial differential equations in the time domain [1]. Integral transforms have become an important working tool of all applied engineers and scientist. The Aboodh transform is a new integral transform similar to the Laplace transform and other integral transforms that are defined in the time domain. The solutions of initial and boundary value problems are given by numerous integral transforms methods. In previous years, numerous notice has been given to deal with the double integral transform, for instance, see [3, 4, 8, 9], and so on which have many applications in various fields of mathematical sciences and engineering such as acoustics, physics, chemistry, etc.,. Many researchers have turned their attention to solve partial differential equation and to develop new methods for solving such equations. Due to the rapid development in the physical science and engineering models [2]. We applied new double Laplace-Aboodh transform to solve Laplace, Poisson, Wave and Heat equations. The main objective of this paper is to present a new method for solving some linear partial differential equations subject to the initial and boundary conditions called double Laplace-Aboodh transform.

1.1. **Definition.** The Laplace transform of the continuous function $f(x)$ is defined by

$$\mathcal{L}[\phi(\zeta)] = \int_0^\infty e^{-\rho\zeta}\phi(\zeta)d\zeta = \Phi(\rho). \tag{1.1}$$

where $\mathcal{L}$ is the Laplace operator. Provided that the integral exists. If the integral is convergent for some value of ρ, then the Laplace transformation of $\phi(\zeta)$ exists

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otherwise not [14].
The inverse Laplace transform is defined by

$$\mathcal{L}^{-1}[\Phi(\rho)] = \phi(\zeta) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\rho\zeta}\Phi(\rho)d\rho, \tag{1.2}$$

where α is a real constant.

1.2. **Definition.** Let $\phi(\tau)$ be an exponential order function in the set

$$\mathcal{H} = \Big\{\phi(\tau) : \exists\mathcal{B}, \alpha_1, \alpha_2 > 0, |\phi(\tau)| < \mathcal{B}e^{|\tau|\alpha_i},\ for\ \tau \in (-1)^i \times [0,\infty),\ i = 1,2\Big\}.$$

where $\mathcal{B}$ is a finite number and α_1, α_2 may finite or infinite [13]. Then the Aboodh transform of the function $\phi(\tau)$ is given by

$$\mathcal{A}[\phi(\tau)] = \Phi(\lambda) = \frac{1}{\lambda}\int_0^\infty e^{-\lambda\tau}\phi(\tau)d\tau,\ \alpha_1 < \tau < \alpha_2, \tag{1.3}$$

where $\mathcal{A}$ is called the Aboodh transform operator.
The inverse Aboodh transform is given by

$$\mathcal{A}^{-1}[\Phi(\lambda)] = \phi(\tau) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \lambda e^{\lambda\tau}\Phi(\lambda)d\lambda; \beta \geq 0. \tag{1.4}$$

where β is a real constant.
In the next definition, we introduce the double Laplace-Aboodh transform.

1.3. **Definition.** The double Laplace-Aboodh transform of the function ϕ of two variables $\zeta, \tau > 0$ is denoted by $\mathcal{L}_\zeta\mathcal{A}_\tau[\phi(\zeta,\tau)] = \Phi(\rho,\lambda)$ and defined by

$$\mathcal{L}_\zeta\mathcal{A}_\tau[\phi(\zeta,\tau)] = \Phi(\rho,\lambda) = \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}\phi(\zeta,\tau)d\zeta d\tau, \tag{1.5}$$

provided the integral exists.

1.4. **Definition.** The inverse double Laplace-Aboodh transform of the function $\phi(\zeta,\tau)$ is defined by

$$\phi(\zeta,\tau) = \mathcal{L}_\zeta^{-1}\mathcal{A}_\tau^{-1}[\Phi(\rho,\lambda)] = \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\rho\zeta}\Big(\int_{\beta-i\infty}^{\beta+i\infty}\lambda e^{\lambda\tau}\Phi(\rho,\lambda)d\lambda\Big)d\rho, \tag{1.6}$$

where α and β are real constants.

2. Existence and uniqueness of the double Laplace-Aboodh transform

2.1. **Definition.** [3] A function $\phi(\zeta,\tau)$ is said to be of exponential orders $\alpha, \beta > 0$, on $0 \leq \zeta, \tau < \infty$, if there exists positive constants K, X and Y such that

$$|\phi(\zeta,\tau)| \leq Ke^{(\alpha\zeta+\beta\tau)},\ for\ all\ \zeta > X,\ \tau > Y,$$

and we write

$$\phi(\zeta,\tau) = o(e^{\alpha\zeta+\beta\tau})\ as\ \zeta,\tau \to \infty.$$

Or, equivalently,

$$\lim_{\zeta\to\infty,\tau\to\infty} e^{-(\rho\zeta+\lambda\tau)}|\phi(\zeta,\tau)| \leq K\lim_{\zeta\to\infty,\tau\to\infty} e^{-(\rho-\alpha)\zeta}e^{-(\lambda-\beta)\tau} = 0,\ \rho > \alpha,\ \lambda > \beta.$$

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2.2. **Theorem.** [8] Let $\phi(\zeta,\tau)$ be a continuous function in every finite intervals $(0,X)$ and $(0,Y)$ and of exponential order $e^{(\alpha\zeta+\beta\tau)}$, then the double Laplace-Aboodh transform of $\phi(\zeta,\tau)$ exists for all $\rho>\alpha$ and $\lambda>\beta$.
Proof. Let $\phi(\zeta,\tau)$ be of exponential order $e^{(\alpha\zeta+\beta\tau)}$, such that

$$\Big|\phi(\zeta,\tau)\Big|\leq Ke^{(\alpha\zeta+\beta\tau)},\ for\ all\ \zeta>X,\ \tau>Y.$$

Then, we have

$$\begin{aligned}\Big|\Phi(\rho,\lambda)\Big| &= \Big|\frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}\phi(\zeta,\tau)d\zeta d\tau\Big|\\ &\leq \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}|\phi(\zeta,\tau)|d\zeta d\tau\\ &\leq \frac{K}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}e^{(\alpha\zeta+\beta\tau)}d\zeta d\tau\\ &= \frac{K}{\lambda}\int_0^\infty e^{-(\rho-\alpha)\zeta}d\zeta\int_0^\infty e^{-(\lambda-\beta)\tau}d\tau\\ &= \frac{K}{(\rho-\alpha)(\lambda^2-\beta\lambda)}.\end{aligned}$$

Thus, the proof is complete.

2.3. **Theorem.** Let $\Phi_1(\rho,\lambda)$ and $\Phi_1(\rho,\lambda)$ be the double Laplace-Aboodh transform of the continuous functions $\phi_1(\zeta,\tau)$ and $\phi_2(\zeta,\tau)$ defined for $\zeta,\tau\geq 0$ respectively. If $\Phi_1(\rho,\lambda)=\Phi_2(\rho,\lambda)$, then $\phi_1(\zeta,\tau)=\phi_2(\zeta,\tau)$.
Proof. Assume that ω_1 and ω_2 are adequately large, since

$$\phi(\zeta,\tau)=\mathcal{L}_\zeta^{-1}\mathcal{A}_\tau^{-1}[\Phi(\rho,\lambda)]=\frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\rho\zeta}\Big(\int_{\beta-i\infty}^{\beta+i\infty}\lambda e^{\lambda\tau}\Phi(\rho,\lambda)d\lambda\Big)d\rho,$$

we deduce that

$$\begin{aligned}\phi_1(\zeta,\tau) &= \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\rho\zeta}\Big(\int_{\beta-i\infty}^{\beta+i\infty}\lambda e^{\lambda\tau}\Phi_1(\rho,\lambda)d\lambda\Big)d\rho\\ &= \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\rho\zeta}\Big(\int_{\beta-i\infty}^{\beta+i\infty}\lambda e^{\lambda\tau}\Phi_2(\rho,\lambda)d\lambda\Big)d\rho\\ &= \phi_2(\zeta,\tau).\end{aligned}$$

This proves the uniqueness of the double Laplace-Aboodh transform.

3. Some Useful Properties of Laplace-Aboodh Transform

3.1. **Linearity property.** If the double Laplace-Aboodh transform of functions $\phi_1(\zeta,\tau)$ and $\phi_2(\zeta,\tau)$ are $\Phi_1(\rho,\lambda)$ and $\Phi_2(\rho,\lambda)$ respectively, then double Laplace-Aboodh transform of $\alpha\phi_1(\zeta,\tau)+\beta\phi_2(\zeta,\tau)$ is given by $\alpha\Phi_1(\rho,\lambda)+\beta\Phi_2(\rho,\lambda)$, where α and β are arbitrary constants.

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Proof.

$$\begin{aligned}\mathcal{L}_\zeta\mathcal{A}_\tau[\alpha\phi_1(\zeta,\tau)+\beta\phi_2(\zeta,\tau)] &= \int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}\Big(\alpha\phi_1(\zeta,\tau)+\beta\phi_2(\zeta,\tau)\Big)d\zeta d\tau\\ &= \alpha\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}\phi_1(\zeta,\tau)d\zeta d\tau\\ &+ \beta\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}\phi_2(\zeta,\tau)d\zeta d\tau\\ &= \alpha\Phi_1(\rho,\lambda)+\beta\Phi_2(\rho,\lambda) \qquad (3.1)\end{aligned}$$

3.2. **Change of scale property.** Let $\phi(\zeta,\tau)$ be a function such that

$$\mathcal{L}_\zeta\mathcal{A}_\tau[\phi(\zeta,\tau)]=\Phi(\rho,\lambda).$$

Then for α and β are positive constants, we have

$$\mathcal{L}_\zeta\mathcal{A}_\tau[\phi(\alpha\zeta,\beta\tau)]=\frac{1}{\alpha\beta}\Phi(\frac{\rho}{\alpha},\frac{\lambda}{\beta}). \qquad (3.2)$$

Proof.

$$\mathcal{L}_\zeta\mathcal{A}_\tau[\phi(\alpha\zeta,\beta\tau)]=\frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}\phi(\alpha\zeta,\beta\tau)d\zeta d\tau.$$

Let $u=\alpha\zeta,\ v=\beta\tau$, then

$$\begin{aligned}\mathcal{L}_\zeta\mathcal{A}_\tau[\phi(u,v)] &= \frac{1}{\alpha\beta\lambda}\int_0^\infty\int_0^\infty e^{-(\frac{\rho}{\alpha}u+\frac{\lambda}{\beta}v)}\phi(u,v)dudv\\ &= \frac{1}{\alpha\beta}\Phi(\frac{\rho}{\alpha},\frac{\lambda}{\beta}).\end{aligned}$$

3.3. **Shifting property.** If $\mathcal{L}_\zeta\mathcal{A}_\tau[\phi(\zeta,\tau)]=\Phi(\rho,\lambda)$, then for any pair of real constants $\alpha,\beta>0$

$$\mathcal{L}_\zeta\mathcal{A}_\tau[e^{(\alpha\zeta+\beta\tau)}\phi(\zeta,\tau)]=\Phi(\rho-\alpha,\lambda-\beta). \qquad (3.3)$$

Proof. Using the definition of double Laplace-Aboodh transform, we get

$$\begin{aligned}\mathcal{L}_\zeta\mathcal{A}_\tau[e^{(\alpha\zeta+\beta\tau)}\phi(\zeta,\tau)] &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}e^{(\alpha\zeta+\beta\tau)}\phi(\zeta,\tau)d\zeta d\tau\\ &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-\big((\rho-\alpha)\zeta+(\lambda-\beta)\tau\big)}\phi(\zeta,\tau)d\zeta d\tau\\ &= \Phi(\rho-\alpha,\lambda-\beta).\end{aligned}$$

3.4. **Derivatives properties.** If $\mathcal{L}_\zeta\mathcal{A}_\tau[\phi(\zeta,\tau)]=\Phi(\rho,\lambda)$, then

$$(1).\ \mathcal{L}_\zeta\mathcal{A}_\tau\Big[\frac{\partial\phi(\zeta,\tau)}{\partial\zeta}\Big]=\rho\Phi(\rho,\lambda)-\mathcal{A}[\phi(0,\tau)]. \qquad (3.4)$$

Proof.

$$\begin{aligned}\mathcal{L}_\zeta\mathcal{A}_\tau\Big[\frac{\partial\phi(\zeta,\tau)}{\partial\zeta}\Big] &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}\frac{\partial\phi(\zeta,\tau)}{\partial\zeta}d\zeta d\tau\\ &= \frac{1}{\lambda}\int_0^\infty e^{-\lambda\tau}d\tau\Big(\int_0^\infty e^{-\rho\zeta}\phi_\zeta(\zeta,\tau)d\zeta\Big).\end{aligned}$$

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Using integration by parts, let $u = e^{-\rho\zeta},\ dv = \phi_\zeta(\zeta,\tau)d\zeta$, then we obtain

$$\begin{aligned}\mathcal{L}_\zeta\mathcal{A}_\tau\Big[\frac{\partial\phi(\zeta,\tau)}{\partial\zeta}\Big] &= \frac{1}{\lambda}\int_0^\infty e^{-\lambda\tau}d\tau\Big(-\phi(0,\tau)+\rho\int_0^\infty e^{-\rho\zeta}\phi(\zeta,\tau)d\zeta\Big)\\ &= \rho\Phi(\rho,\lambda)-\mathcal{A}[\phi(0,\tau)].\end{aligned}$$

$$(2).\ \mathcal{L}_\zeta\mathcal{A}_\tau\Big[\frac{\partial\phi(\zeta,\tau)}{\partial\tau}\Big]=\lambda\Phi(\rho,\lambda)-\frac{1}{\lambda}\mathcal{L}[\phi(\zeta,0)]. \tag{3.5}$$

Proof.

$$\begin{aligned}\mathcal{L}_\zeta\mathcal{A}_\tau\Big[\frac{\partial\phi(\zeta,\tau)}{\partial\tau}\Big] &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}\frac{\partial\phi(\zeta,\tau)}{\partial\tau}d\zeta d\tau\\ &= \frac{1}{\lambda}\int_0^\infty e^{-\rho\zeta}d\zeta\Big(\int_0^\infty e^{-\lambda\tau}\phi_\tau(\zeta,\tau)d\tau\Big).\end{aligned}$$

Using integration by parts, let $u = e^{-\lambda\tau},\ dv = \phi_\tau(\zeta,\tau)d\tau$, then we obtain

$$\begin{aligned}\mathcal{L}_\zeta\mathcal{A}_\tau\Big[\frac{\partial\phi(\zeta,\tau)}{\partial\tau}\Big] &= \frac{1}{\lambda}\int_0^\infty e^{-\rho\zeta}d\zeta\Big(-\phi(\zeta,0)+\lambda\int_0^\infty e^{-\lambda\tau}\phi(\zeta,\tau)d\tau\Big)\\ &= \lambda\Phi(\rho,\lambda)-\frac{1}{\lambda}\mathcal{L}[\phi(\zeta,0)].\end{aligned}$$

$$(3).\ \mathcal{L}_\zeta\mathcal{A}_\tau\Big[\frac{\partial^2\phi(\zeta,\tau)}{\partial\zeta^2}\Big]=\rho^2\Phi(\rho,\lambda)-\rho\mathcal{A}[\phi(0,\tau)]-\mathcal{A}[\phi_\zeta(0,\tau)]. \tag{3.6}$$

Proof.

$$\begin{aligned}\mathcal{L}_\zeta\mathcal{A}_\tau\Big[\frac{\partial^2\phi(\zeta,\tau)}{\partial\zeta^2}\Big] &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}\frac{\partial^2\phi(\zeta,\tau)}{\partial\zeta^2}d\zeta d\tau\\ &= \frac{1}{\lambda}\int_0^\infty e^{-\lambda\tau}d\tau\Big(\int_0^\infty e^{-\rho\zeta}\phi_{\zeta\zeta}(\zeta,\tau)d\zeta\Big).\end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned}\mathcal{L}_\zeta\mathcal{A}_\tau\Big[\frac{\partial^2\phi(\zeta,\tau)}{\partial\zeta^2}\Big] &= \frac{1}{\lambda}\int_0^\infty e^{-\lambda\tau}d\tau\Big(-\phi_\zeta(0,\tau)+\rho\Big\{-\phi(0,\tau)+\rho\int_0^\infty e^{-\rho\zeta}\phi(\zeta,\tau)d\zeta\Big\}\Big)\\ &= \rho^2\Phi(\rho,\lambda)-\rho\mathcal{A}[\phi(0,\tau)]-\mathcal{A}[\phi_\zeta(0,\tau)].\end{aligned}$$

Similarly, we can prove that:

$$(4).\ \mathcal{L}_\zeta\mathcal{A}_\tau\Big[\frac{\partial^2\phi(\zeta,\tau)}{\partial\tau^2}\Big]=\lambda^2\Phi(\rho,\lambda)-\mathcal{L}[\phi(\zeta,0)]-\frac{1}{\lambda}\mathcal{L}[\phi_\tau(\zeta,0)].$$

$$(5).\ \mathcal{L}_\zeta\mathcal{A}_\tau\Big[\frac{\partial^2\phi(\zeta,\tau)}{\partial\zeta\partial\tau}\Big]=\rho\lambda\Phi(\rho,\lambda)-\frac{\rho}{\lambda}\mathcal{L}[\phi(\zeta,0)]-\mathcal{A}[\phi_\tau(0,\tau)].$$

4. Convolution theorem of double Laplace-Aboodh transform

4.1. **Definition.** The convolution of the functions $\phi(\zeta,\tau)$ and $\psi(\zeta,\tau)$ is denoted by $(\phi**\psi)(\zeta,\tau)$ and defined by

$$(\phi**\psi)(\zeta,\tau)=\int_0^\infty\int_0^\infty\phi(\zeta-\varepsilon,\tau-\delta)\psi(\varepsilon,\delta)d\varepsilon d\delta. \tag{4.1}$$

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4.2. **Theorem.** [3] If $\Phi(\rho,\lambda)=\mathcal{L}_\zeta\mathcal{A}_\tau[\phi(\zeta,\tau)]$, then for constants ε,δ we have

$$\mathcal{L}_\zeta\mathcal{A}_\tau[\phi(\zeta-\varepsilon,\tau-\delta)H(\zeta-\varepsilon,\tau-\delta)]=e^{-(\rho\varepsilon+\lambda\delta)}\Phi(\rho,\lambda), \tag{4.2}$$

where $H(\zeta,\tau)$ is the Heaviside unit step function defined by

$$H(\zeta-\varepsilon,\tau-\delta)=\begin{cases}1, & \zeta>\varepsilon,\tau>\delta,\\ 0, & otherwis.\end{cases} \tag{4.3}$$

Proof. Using definition of double Laplace-Aboodh transform, we have

$$\begin{aligned}\mathcal{L}_\zeta\mathcal{A}_\tau[\phi(\zeta-\varepsilon,\tau-\delta)H(\zeta-\varepsilon,\tau-\delta)] &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}\phi(\zeta-\varepsilon,\tau-\delta)H(\zeta-\varepsilon,\tau-\delta)d\zeta d\tau\\ &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}\phi(\zeta-\varepsilon,\tau-\delta)d\zeta d\tau, \qquad (4.4)\end{aligned}$$

by putting $\zeta-\varepsilon=\vartheta$, $\tau-\delta=\upsilon$, then, we have

$$\begin{aligned}\mathcal{L}_\zeta\mathcal{A}_\tau[\phi(\zeta-\varepsilon,\tau-\delta)H(\zeta-\varepsilon,\tau-\delta)] &= e^{-(\rho\varepsilon+\lambda\delta)}\frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\vartheta+\lambda\upsilon)}\phi(\vartheta,\upsilon)d\vartheta d\upsilon\\ &= e^{-(\rho\varepsilon+\lambda\delta)}\Phi(\rho,\lambda). \qquad (4.5)\end{aligned}$$

4.3. **Theorem.** Let $\Phi(\rho,\lambda)$ and $\Psi(\rho,\lambda)$ be the double Laplace-Aboodh transform of the functions $\phi(\zeta,\tau)$ and $\psi(\zeta,\tau)$ respectively, then

$$\mathcal{L}_\zeta\mathcal{A}_\tau[(\phi**\psi)(\zeta,\tau)]=\lambda\Phi(\rho,\lambda)\Psi(\rho,\lambda).$$

Proof. By definition (1.3), we have

$$\begin{aligned}\mathcal{L}_\zeta\mathcal{A}_\tau[(\phi**\psi)(\zeta,\tau)] &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}(\phi**\psi)(\zeta,\tau)d\zeta d\tau\\ &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}\Big\{\int_0^\infty\int_0^\infty\phi(\zeta-\varepsilon,\tau-\delta)\psi(\varepsilon,\delta)d\varepsilon d\delta\Big\}d\zeta d\tau\end{aligned}$$

Using the Heaviside unit step function

$$\begin{aligned}&= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}\Big\{\int_0^\infty\int_0^\infty\phi(\zeta-\varepsilon,\tau-\delta)H(\zeta-\varepsilon,\tau-\delta)\psi(\varepsilon,\delta)d\varepsilon d\delta\Big\}d\zeta d\tau\\ &= \int_0^\infty\int_0^\infty\psi(\varepsilon,\delta)d\varepsilon d\delta\Big\{\frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}\phi(\zeta-\varepsilon,\tau-\delta)H(\zeta-\varepsilon,\tau-\delta)d\zeta d\tau\Big\}.\end{aligned}$$

By Theorem (4.2)

$$\begin{aligned}&= \int_0^\infty\int_0^\infty\psi(\varepsilon,\delta)d\varepsilon d\delta\Big\{e^{-(\rho\varepsilon+\lambda\delta)}\Phi(\rho,\lambda)\Big\}\\ &= \Phi(\rho,\lambda)\int_0^\infty\int_0^\infty e^{-(\rho\varepsilon+\lambda\delta)}\psi(\varepsilon,\delta)d\varepsilon d\delta\\ &= \lambda\Phi(\rho,\lambda)\Psi(\rho,\lambda).\end{aligned}$$

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5. The Double Laplace-Aboodh Transform of Some Elementary Functions

(1). If the function $\phi(\zeta,\tau)=1$, then

$$\mathcal{L}_{\zeta}\mathcal{A}_{\tau}[\phi(\zeta,\tau)]=\frac{1}{\lambda}\int_0^{\infty}\int_0^{\infty}e^{-(\rho\zeta+\lambda\tau)}d\zeta d\tau=\frac{1}{\rho\lambda^2}. \tag{5.1}$$

(2). If the function $\phi(\zeta,\tau)=\zeta\tau$, then

$$\mathcal{L}_{\zeta}\mathcal{A}_{\tau}[\phi(\zeta,\tau)]=\frac{1}{\lambda}\int_0^{\infty}\int_0^{\infty}e^{-(\rho\zeta+\lambda\tau)}\zeta\tau d\zeta d\tau=\frac{1}{\rho^2\lambda^3}. \tag{5.2}$$

(3). If the function $\phi(\zeta,\tau)=\zeta^2\tau^2$, then

$$\mathcal{L}_{\zeta}\mathcal{A}_{\tau}[\phi(\zeta,\tau)]=\frac{1}{\lambda}\int_0^{\infty}\int_0^{\infty}e^{-(\rho\zeta+\lambda\tau)}\zeta^2\tau^2 d\zeta d\tau=\frac{4}{\rho^3\lambda^4}. \tag{5.3}$$

(4). If the function $\phi(\zeta,\tau)=\zeta^n\tau^m,\ n,m=0,1,2,\ \dots$, then

$$\mathcal{L}_{\zeta}\mathcal{A}_{\tau}[\phi(\zeta,\tau)] = \frac{1}{\lambda}\int_0^{\infty}\int_0^{\infty}e^{-(\rho\zeta+\lambda\tau)}\zeta^n\tau^m d\zeta d\tau=\frac{n!m!}{\rho^{n+1}\lambda^{m+2}}. \tag{5.4}$$

(5). If the function $\phi(\zeta,\tau)=\zeta^{\sigma}\tau^{\nu},\ \sigma\geq -1,\ \nu\geq -1$, then

$$\mathcal{L}_{\zeta}\mathcal{A}_{\tau}[\phi(\zeta,\tau)]=\frac{1}{\lambda}\int_0^{\infty}\int_0^{\infty}e^{-(\rho\zeta+\lambda\tau)}\zeta^{\sigma}\tau^{\nu}d\zeta d\tau = \int_0^{\infty}e^{-\rho\zeta}\zeta^{\sigma}d\zeta\int_0^{\infty}\frac{1}{\lambda}e^{-\lambda\tau}\tau^{\nu}d\tau,$$

let $x=\rho\zeta$ and $y=\lambda\tau$

$$\begin{aligned}\mathcal{L}_{\zeta}\mathcal{A}_{\tau}[\phi(\zeta,\tau)] &= \frac{1}{\rho^{\sigma+1}}\int_0^{\infty}e^{-x}x^{\sigma}dx\Big(\frac{1}{\lambda^{\nu+2}}\int_0^{\infty}e^{-y}y^{\nu}dy\Big)\\ &= \Gamma(\sigma+1)\Big(\frac{1}{\rho^{\sigma+1}}\Big)\Gamma(\nu+1)\frac{1}{\lambda^{\nu+2}}.\end{aligned} \tag{5.5}$$

Where, $\Gamma(.)$ is the Euler gamma function.
(6). If the function $\phi(\zeta,\tau)=e^{(\alpha\zeta+\beta\tau)},\ \alpha,\beta=0,1,2,\ \dots$, then

$$\mathcal{L}_{\zeta}\mathcal{A}_{\tau}[\phi(\zeta,\tau)]=\frac{1}{\lambda}\int_0^{\infty}\int_0^{\infty}e^{-(\rho\zeta+\lambda\tau)}e^{(\alpha\zeta+\beta\tau)}d\zeta d\lambda=\frac{1}{(\rho-\alpha)(\lambda^2-\beta\lambda)}. \tag{5.6}$$

Similarly,

$$\begin{aligned}\mathcal{L}_{\zeta}\mathcal{A}_{\tau}[e^{i(\alpha\zeta+\beta\tau)}] &= \frac{1}{\lambda}\int_0^{\infty}\int_0^{\infty}e^{-(\rho\zeta+\lambda\tau)}e^{i(\alpha\zeta+\beta\tau)}d\zeta d\tau=\frac{1}{\lambda(\rho-i\alpha)}\frac{1}{(\lambda-i\beta)}\\ &= \frac{(\rho\lambda-\alpha\beta)+i(\rho\beta+\alpha\lambda)}{(\rho^2+\alpha^2)(\lambda^3+\beta^2\lambda)}.\end{aligned} \tag{5.7}$$

Consequently,

$$\mathcal{L}_{\zeta}\mathcal{A}_{\tau}[\cos(\alpha\zeta+\beta\tau)]=\frac{\rho\lambda-\alpha\beta}{(\rho^2+\alpha^2)(\lambda^3+\beta^2\lambda)},$$

$$\mathcal{L}_{\zeta}\mathcal{A}_{\tau}[\sin(\alpha\zeta+\beta\tau)]=\frac{\rho\beta+\alpha\lambda}{(\rho^2+\alpha^2)(\lambda^3+\beta^2\lambda)}.$$

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(7). If the function $\phi(\zeta,\tau)=\cosh(\alpha\zeta+\beta\tau),\ \alpha,\beta=0,1,2,\ \ldots$, then

$$\begin{aligned}\mathcal{L}_\zeta\mathcal{A}_\tau[\phi(\zeta,\tau)] &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}\cosh(\alpha\zeta+\beta\tau)d\zeta d\tau\\ &= \frac{\rho\lambda+\alpha\beta}{(\rho^2-\alpha^2)(\lambda^3-\beta^2\lambda)}. \end{aligned}\tag{5.8}$$

(8). If the function $\phi(\xi,\eta)=\sinh(\alpha\zeta+\beta\tau),\ \alpha,\beta=0,1,2,\ \ldots$, then

$$\begin{aligned}\mathcal{L}_\zeta\mathcal{A}_\tau[\phi(\zeta,\tau)] &= \frac{1}{\lambda}\int_0^\infty\int_0^\infty e^{-(\rho\zeta+\lambda\tau)}\sinh(\alpha\zeta+\beta\tau)d\zeta d\lambda\\ &= \frac{\rho\beta+\alpha\lambda}{(\rho^2-\alpha^2)(\lambda^3-\beta^2\lambda)}. \end{aligned}\tag{5.9}$$

6. Applications

In this section, to establish the efficiency of the suggestion method we consider second- order linear partial differential equations with initial and boundary problems. Let the second-order nonhomogeneous linear partial differential equation in two independent variables (ζ,τ) be in the form:

$$A\phi_{\zeta\zeta}(\zeta,\tau)+B\phi_{\tau\tau}(\zeta,\tau)+C\phi_\zeta(\zeta,\tau)+D\phi_\tau(\zeta,\tau)+E\phi(\zeta,\tau)=h(\zeta,\tau),\ (\zeta,\tau)\in\mathbb{R}^2_+,\tag{6.1}$$

with the initial conditions:

$$\phi(\zeta,0)=T_1(\zeta),\ \phi_\tau(\zeta,0)=T_2(\zeta),\tag{6.2}$$

and the boundary conditions:

$$\phi(0,\tau)=T_3(\tau),\ \phi_\zeta(0,\tau)=T_4(\tau),\tag{6.3}$$

where A,B,C,D and E are constants and $h(\zeta,\tau)$ is the source term.
Using the property of partial derivative of the double Laplace-Aboodh transform for equation (6.1), single Laplace transform for equation (6.2) and single Aboodh transform for equation (6.3) and simplifying, we obtain that:

$$\Phi(\rho,\lambda)=\frac{(B+\frac{D}{\lambda})T_1(\rho)+\frac{B}{\lambda}T_2(\rho)+(A\rho+C)T_3(\lambda)+AT_4(\lambda)+H(\rho,\lambda)}{(A\rho^2+B\lambda^2+C\rho+D\lambda+E)},\tag{6.4}$$

where $H(\rho,\lambda)=\mathcal{L}_\zeta\mathcal{A}_\tau[h(\zeta,\tau)]$.
Lastly, solving this algebraic equation in $\Phi(\rho,\lambda)$ and taking the inverse double Laplace-Aboodh transform on both sides of equation (6.4), yields

$$\phi(\zeta,\tau)=\mathcal{L}_\zeta^{-1}\mathcal{A}_\tau^{-1}\Big[\frac{(B+\frac{D}{\lambda})T_1(\rho)+\frac{B}{\lambda}T_2(\rho)+(A\rho+C)T_3(\lambda)+AT_4(\lambda)+H(\rho,\lambda)}{(A\rho^2+B\lambda^2+C\rho+D\lambda+E)}\Big],\tag{6.5}$$

which represent the general formula for the solution of equation (6.1) by double Laplace-Aboodh transform method.

Example 6.1. *Consider the following boundary Laplace equation*

$$\phi_{\zeta\zeta}(\zeta,\tau)+\phi_{\tau\tau}(\zeta,\tau)=0,\qquad (\zeta,\tau)\in\mathbb{R}^2_+,\tag{6.6}$$

with the conditions:

$$\begin{cases}\phi(\zeta,0)=\sinh\zeta=T_1(\zeta), & \phi_\tau(\zeta,0)=0=T_2(\zeta),\\ \phi(0,\tau)=0=T_3(\tau), & \phi_\zeta(0,\tau)=\cos\tau=T_4(\tau).\end{cases}$$

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Solution:
Substituting

$$T_1(\rho)=\frac{1}{\rho^2-1},\ T_2(\rho)=0,\ T_3(\lambda)=0,\ T_4(\lambda)=\frac{1}{\lambda^2+1},\ H(\rho,\lambda)=0,$$

in (6.5) and simplifying, we get a solution of (6.6)

$$\phi(\zeta,\tau) = \mathcal{L}_\zeta^{-1}\mathcal{A}_\tau^{-1}\Big[\frac{1}{\rho^2+\lambda^2}\ \Big(\frac{1}{\rho^2-1}+\frac{1}{\lambda^2+1}\Big)\Big]=\sinh\zeta\cos\tau. \qquad (6.7)$$

Example 6.2. *Consider the following boundary Poisson equation*

$$\phi_{\zeta\zeta}(\zeta,\tau)+\phi_{\tau\tau}(\zeta,\tau)=2e^{-\zeta+\tau}, \qquad (\zeta,\tau)\in\mathbb{R}_+^2, \qquad (6.8)$$

with the conditions:

$$\begin{cases}\phi(\zeta,0)=e^{-\zeta}+\cos\zeta=T_1(\zeta), & \phi_\tau(\zeta,0)=e^{-\zeta}+\cos\zeta=T_2(\zeta),\\ \phi(0,\tau)=2e^\tau=T_3(\tau), & \phi_\zeta(0,\tau)=-e^\tau=T_4(\tau).\end{cases}$$

Solution:
Substituting

$$\begin{cases}T_1(\rho)=\frac{1}{\rho+1}+\frac{\rho}{\rho^2+1}, & T_2(\rho)=\frac{1}{\rho+1}+\frac{\rho}{\rho^2+1},\\ T_3(\lambda)=\frac{2}{\lambda(\lambda-1)}, & T_4(\lambda)=\frac{-1}{\lambda(\lambda-1)},\\ H(\rho,\lambda)=\frac{2}{\lambda(\rho+1)(\lambda-1)}, & \end{cases}$$

in (6.4) and simplifying, we get

$$\begin{aligned}\Phi(\rho,\lambda) &= \frac{\Big(\frac{2}{\lambda(\rho+1)(\lambda-1)}+\frac{2\rho}{\lambda(\lambda-1)}-\frac{1}{\lambda(\lambda-1)}+\frac{1}{\rho+1}+\frac{\rho}{\rho^2+1}+\frac{1}{\lambda(\rho+1)}+\frac{\rho}{\lambda(\rho^2+1)}\Big)}{(\rho^2+\lambda^2)}\\ &= \frac{1}{\lambda(\rho+1)(\lambda-1)}+\frac{\rho}{\lambda(\rho^2+1)(\lambda-1)}. \qquad (6.9)\end{aligned}$$

Taking the inverse double Laplace-Aboodh transform of equation (6.9), we get a solution of (6.8)

$$\begin{aligned}\phi(\zeta,\tau) &= \mathcal{L}_\zeta^{-1}\mathcal{A}_\tau^{-1}\Big[\frac{1}{\lambda(\rho+1)(\lambda-1)}+\frac{\rho}{\lambda(\rho^2+1)(\lambda-1)}\Big]\\ &= e^{-\zeta+\tau}+e^\tau\cos\zeta. \qquad (6.10)\end{aligned}$$

Example 6.3. *Consider the following nonhomogeneous Wave equation*

$$\phi_{\tau\tau}(\zeta,\tau)=\phi_{\zeta\zeta}(\zeta,\tau)+6\tau+2\zeta, \qquad (\zeta,\tau)\in\mathbb{R}_+^2, \qquad (6.11)$$

with the conditions:

$$\begin{cases}\phi(\zeta,0)=0=T_1(\zeta), & \phi_\tau(\zeta,0)=\sin\zeta=T_2(\zeta),\\ \phi(0,\tau)=\tau^3=T_3(\tau), & \phi_\zeta(0,\tau)=\tau^2+\sin\tau=T_4(\tau).\end{cases}$$

Solution:
Substituting

$$\begin{cases}T_1(\rho)=0, & T_2(\rho)=\frac{1}{\rho^2+1},\\ T_3(\lambda)=\frac{6}{\lambda^5}, & T_4(\lambda)=\frac{2}{\lambda^4}+\frac{1}{\lambda(\lambda^2+1)},\\ H(\rho,\lambda)=\frac{6}{\rho\lambda^3}+\frac{2}{\rho^2\lambda^2}, & \end{cases}$$

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in (6.4) and simplifying, we get

$$\begin{aligned}\Phi(\rho,\lambda) &= \frac{1}{\rho^2-\lambda^2}\Big(\frac{6\rho}{\lambda^5}+\frac{2}{\lambda^4}+\frac{1}{\lambda(\lambda^2+1)}-\frac{1}{\lambda(\rho^2+1)}-\frac{2}{\rho^2\lambda^2}-\frac{6}{\rho\lambda^3}\Big)\\ &= \frac{6}{\rho\lambda^5}+\frac{2}{\rho^2\lambda^4}+\frac{1}{\lambda(\lambda^2+1)(\rho^2+1)}. \qquad (6.12)\end{aligned}$$

Taking the inverse double Laplace-Aboodh transform of equation (6.12), we get a solution of (6.11)

$$\begin{aligned}\phi(\zeta,\tau) &= \mathcal{L}_\zeta^{-1}\mathcal{A}_\tau^{-1}\Big[\frac{6}{\rho\lambda^5}+\frac{2}{\rho^2\lambda^4}+\frac{1}{\lambda(\lambda^2+1)(\rho^2+1)}\Big]\\ &= \tau^3+\zeta\tau^2+\sin\zeta\sin\tau. \qquad (6.13)\end{aligned}$$

Example 6.4. *Consider the following nonhomogeneous Heat equation*

$$\phi_\tau(\zeta,\tau)=\phi_{\zeta\zeta}(\zeta,\tau)-\phi(\zeta,\tau)+1, \qquad (\zeta,\tau)\in\mathbb{R}_+^2, \qquad (6.14)$$

with the conditions:

$$\begin{cases}\phi(\zeta,0)=1+\sin\zeta=T_1(\zeta), & \phi_\tau(\zeta,0)=-2\sin\zeta=T_2(\zeta),\\ \phi(0,\tau)=1=T_3(\tau), & \phi_\zeta(0,\tau)=e^{-2\tau}=T_4(\tau).\end{cases}$$

Solution:
Substituting

$$\begin{cases}T_1(\rho)=\frac{1}{\rho}+\frac{1}{\rho^2+1}, & T_2(\rho)=\frac{-2}{\rho^2+1},\\ T_3(\lambda)=\frac{1}{\lambda^2}, & T_4(\lambda)=\frac{1}{\lambda(\lambda+2)},\\ H(\rho,\lambda)=\frac{1}{\rho\lambda^2}, & \end{cases}$$

in (6.4) and simplifying, we get a solution of (6.14)

$$\begin{aligned}\phi(\zeta,\tau) &= \mathcal{L}_\zeta^{-1}\mathcal{A}_\tau^{-1}\Big[\frac{1}{\rho^2-\lambda-1}\Big(\frac{\rho}{\lambda^2}+\frac{1}{\lambda(\lambda+2)}-\frac{1}{\rho\lambda}-\frac{1}{\lambda(\rho^2+1)}-\frac{1}{\rho\lambda^2}\Big)\Big]\\ &= \mathcal{L}_\zeta^{-1}\mathcal{A}_\tau^{-1}\Big[\frac{1}{\rho\lambda^2}+\frac{1}{\lambda(\rho^2+1)(\lambda+2)}\Big]\\ &= 1+e^{-2\tau}\sin\zeta. \qquad (6.15)\end{aligned}$$

conclusion

In conclusion, double Laplace-Aboodh transform is an influential transform among all the integral transforms of exponential sort kernels, the double Laplace-Aboodh transform method for solving partial differential equations is studied. We showed the popular properties and theorems for double Laplace-Aboodh transform and equipped some examples.

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