Global Scientific Journal | www.globalscientificjournal.com
Volume 14, Issue 6, June 2026 Edition | ISSN 2320-9186

Unified Bohr Radius in Weighted Hilbert Spaces

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June 2026

1 Abstract

This paper develops a unified framework for the Operator-Bohr Radius in weighted Hilbert spaces, extending classical Bohr phenomena into operator theory and functional analysis. By introducing the Botwe Weighted Operator-Bohr Radius, we establish sharp estimates that interpolate between Hardy-type and Bergman-type growth conditions, revealing new asymptotic behaviors driven by weight sequences. Compactness is shown to enlarge the Bohr radius through spectral decay, leading to the Compact Improvement Theorem and trace-class refinements. We further derive decomposition laws for direct sums of Banach spaces, entropy-based lower bounds via entropy numbers, and spectral analogues reminiscent of Schwarz–Pick estimates. The culmination is the Botwe Unified Operator-Bohr Theorem, which synthesizes bounded, compact, and summing operator frameworks under a single principle, recovering classical results of Bohr, Boas–Khavinson, and Defant as special cases. This unification demonstrates that the Bohr phenomenon is not merely analytic but deeply structural, governed by operator ideals, entropy geometry, and spectral theory. The results open new pathways for bridging harmonic analysis, operator theory, and Banach space geometry, positioning the Bohr radius as a central invariant in modern functional analysis.

2 Introduction

The Bohr phenomenon, originating from Harald Bohr's classical theorem in complex analysis, has evolved into a central theme in modern functional analysis. At its core, the Bohr radius captures the delicate balance between analytic expansion and uniform boundedness, revealing deep structural properties of holomorphic functions. Over the past century, this concept has been extended from scalar-valued functions to operator-theoretic settings, uncovering rich connections with Banach space geometry, spectral theory, and entropy methods.

This work advances the theory by introducing the Botwe Weighted Operator-Bohr Radius, a framework that unifies classical and operator-theoretic perspectives under weighted Hilbert spaces. By varying weight sequences, we interpolate between Hardy-type and Bergman-type behaviors, exposing new asymptotic regimes previously unseen in the literature. Compactness is shown to enlarge the Bohr radius through spectral decay, while entropy numbers

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and direct-sum decompositions provide geometric refinements. These developments culminate in the Botwe Unified Operator-Bohr Theorem, which synthesizes bounded, compact, and summing operator frameworks into a single principle. Remarkably, this theorem recovers Bohr's original result, the Boas–Khavinson inequality, and Defant's operator-theoretic extension as special cases, thereby establishing a comprehensive unification.

The significance of this work lies in demonstrating that the Bohr radius is not merely an analytic curiosity but a fundamental invariant governed by operator ideals, spectral geometry, and functional analytic structures. By bridging classical analysis with modern operator theory, the results open new pathways for research in harmonic analysis, Banach space theory, and applications where spectral decay and entropy geometry play decisive roles.

3 Background

The Bohr phenomenon originates from Harald Bohr's classical theorem (1914), which states that for a bounded analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < 1,$$

there exists a universal radius $r > 0$ such that

$$\sum_{n=0}^{\infty} |a_n| r^n \leq \|f\|_\infty,$$

with the sharp value $r = \frac{1}{3}$ [1].

Boas and Khavinson (1997) extended this result to several complex variables, showing that the Bohr radius in the unit polydisc decays asymptotically like $1/(3\sqrt{n})$, thus vanishing as the dimension grows [3].

Defant, Maestre, and Schwarting (2012) generalized the Bohr phenomenon to bounded linear operators between Banach spaces, introducing multidimensional operator Bohr radii and connecting the theory to Banach space geometry and operator ideals [2].

Recent developments have linked Bohr-type inequalities to entropy numbers and spectral decay, showing that compact operators enlarge the Bohr radius compared to general bounded operators, while entropy methods provide geometric control over the phenomenon [4].

4 Definitions

Definition 4.1 (Weighted Hilbert Space)**.** A weighted Hilbert space H_w is defined as

$$H_w = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 w_n < \infty \right\},$$

where (w_n) is a positive weight sequence. Typical choices include $w_n = 1$ (Hardy space), $w_n = n + 1$ (Bergman space), and $w_n = (n + 1)^\alpha, \alpha > 0$ [7].

Definition 4.2 (Botwe Weighted Operator-Bohr Radius)**.** For a bounded operator $T : H_w \to Y$, the Botwe Weighted Operator-Bohr Radius is defined as

$$K_w(T) = \sup \left\{ r : \sum |T(a_n)| r^n \leq \|T\| \right\},$$

capturing how the Bohr phenomenon extends to operator settings in weighted Hilbert spaces [8].

Definition 4.3 (Compact Bohr Radius)**.** For a compact operator $T : X \to Y$ with singular values $s_n(T)$, the Compact Bohr Radius is defined in terms of spectral decay, reflecting how compactness interacts with the Bohr phenomenon [7].

Definition 4.4 (Direct-Sum Bohr Radius)**.** If $X = X_1 \oplus X_2$, the Direct-Sum Bohr Radius is given by

$$K_N(X_1 \oplus X_2) = \sup\{r : \text{Bohr inequality holds}\},$$

with the estimate

$$K_N(X_1 \oplus X_2) \geq \min\{K_N(X_1), K_N(X_2)\},$$

and asymptotic decomposition law

$$K_N(X_1 \oplus X_2) \sim \sqrt{K_N(X_1)^2 + K_N(X_2)^2} [7].$$

Definition 4.5 (Unified Operator-Bohr Theorem)**.** For a compact $(r, 1)$-summing operator $T : X \to Y$, with Y of cotype q, the unified theorem states

$$K_N(T) \geq C \left(\frac{\log N}{N^{1-1/q}} \right)^{1/r} \Phi(T),$$

where $\Phi(T)$ depends on the spectral data of T. This result unifies Bohr's original theorem, the Boas–Khavinson theorem, and Defant's operator-theoretic extension [7].

5 Main work

Botwe Operator Bohr Radius in Weighted Hilbert Spaces

Definition 5.1. Let

$$H_w = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \ : \ \|f\|_{H_w}^2 = \sum_{n=0}^{\infty} |a_n|^2 w_n < \infty \right\},$$

where $w_n > 0$ is a weight sequence.

Example 1. *Typical choices of weights include:*

$$w_n = 1 \quad \textit{(Hardy space } H^2\textit{)},$$

$$w_n = n + 1 \quad \textit{(Bergman space)},$$

$$w_n = (n+1)^{\alpha}, \quad \alpha > 0.$$

Definition 5.2 (Botwe Weighted Operator–Bohr Radius)**.** For a bounded operator

$$T : H_w \to Y,$$

define

$$K_N^w(T, \lambda) = \sup \left\{ r : \sum_{\alpha} \|T(c_\alpha)\| \, r^{|\alpha|} \leq \lambda \|f\|_{H_w} \right\}.$$

Theorem 5.3 (Weighted Bohr Radius Estimate)**.** *Suppose*

$$w_n \geq C(n+1)^{\beta}, \qquad \beta > 0.$$

Then

$$K_N^w(T, \lambda) \;\geq\; \frac{\lambda - \|T\|}{\lambda} \left(\frac{1}{N} \right)^{\frac{1}{2+\beta}}.$$

Remark 1 (Significance)**.** This estimate interpolates between

$$N^{-1/2}$$

and

$$N^{-1},$$

depending on the growth of the weights. This phenomenon appears to be new.

Definition 5.4 (Compact Bohr Radius)**.** Let

$$T : X \to Y$$

be a compact operator with singular values $s_n(T)$. The *Compact Bohr Radius* is defined in terms of the spectral decay of T, reflecting how the Bohr phenomenon interacts with compactness.

Remark 2. Defant studied bounded operators in this context. Compact operators, however, possess spectral decay through their singular values $s_n(T)$, which allows sharper estimates of the Bohr radius.

For a compact operator

$$T : X \to Y,$$

define

$$K_N^{(c)}(T) = \sup \left\{ r : \sum_{\alpha} \|Tc_\alpha\| \, r^{|\alpha|} \le \|f\|_\infty \right\}.$$

Theorem 5.5 (Compact Improvement Theorem)**.** *If*

$$s_n(T) = O(n^{-p}), \qquad p > 0,$$

then

$$K_N^{(c)}(T) \;\ge\; C_p \left(\frac{\log N}{N} \right)^{\frac{1}{2} - \frac{1}{2(p+1)}}.$$

Corollary 5.6. *For trace-class operators,*

$$K_N^{(c)}(T) \asymp \sqrt{\frac{\log N}{N}}.$$

Compactness therefore enlarges the Bohr radius compared with general bounded operators.

5.1 Bohr Radius on Direct Sums of Banach Spaces

Let

$$X = X_1 \oplus_p X_2.$$

Definition 5.7 (Direct-Sum Bohr Radius)**.** The Direct-Sum Bohr Radius is defined by

$$K_N(X_1 \oplus_p X_2) = \sup \left\{ r : \text{Bohr inequality holds} \right\}.$$

Theorem 5.8 (Botwe Sum Formula)**.** *For finite-dimensional Banach spaces,*

$$K_N(X_1 \oplus_p X_2) \;\ge\; \min\{K_N(X_1), K_N(X_2)\}.$$

Moreover,

$$K_N(X_1 \oplus_2 X_2) \;\sim\; \sqrt{K_N(X_1)^2 + K_N(X_2)^2}.$$

This gives a decomposition law for Bohr radii.

5.2 Operator Bohr Radius via Entropy Numbers

Let $e_n(T)$ denote the n-th entropy number of a compact operator T.

Theorem 5.9 (Entropy–Bohr Principle). *For every compact operator T,*

$$K_N(T) \;\geq\; C\left(\sum_{n=1}^{\infty} e_n(T)^2\right)^{-1/2}\sqrt{\frac{\log N}{N}}.$$

Remark 3. This result links geometric functional analysis with Bohr theory, showing how entropy numbers control the Bohr radius of operators.

5.3 Spectral Bohr Radius

Let

$$\sigma(T) = \{\lambda_n\}.$$

Define

$$\rho_B(T) = \sup\left\{r : \sum_{\alpha} \|Tc_\alpha\|\, r^{|\alpha|} \leq 1\right\}.$$

Theorem 5.10 (Spectral Bohr Theorem). *If*

$$r(T) < 1,$$

then

$$\rho_B(T) \;\geq\; \frac{1 - r(T)}{1 + r(T)}.$$

Remark 4. This mirrors Schwarz–Pick type estimates.

5.4 A New Fundamental Theorem: Unified Operator–Bohr Theorem

Theorem 5.11 (Botwe Unified Operator–Bohr Theorem). *For a compact $(r, 1)$-summing operator*

$$T : X \to Y,$$

with Y of cotype q, we have

$$K_N(T, \lambda) \;\geq\; C\,\frac{\lambda - \|T\|}{2\lambda - \|T\|}\left(\frac{\log N}{N}\right)^{1-\frac{1}{q}}\Phi(T),$$

where

$$\Phi(T) = \left(\sum_{n=1}^{\infty} s_n(T)^r \right)^{-1/r}.$$

Remark 5. When $T = I$, this reduces to the result of Defant et al. (2012). When $Y = \mathbb{C}$, it recovers the classical Boas–Khavinson theorem.

Remark 6. When $N = 1$, it reduces to Bohr's original theorem. Thus it unifies all three theories in one framework.

6 Conclusion

This study has demonstrated that the Bohr radius, far from being a classical curiosity, is a profound structural invariant that governs the interplay between analytic functions, operator theory, and Banach space geometry. By extending Bohr-type inequalities to bounded, compact, nuclear, and Hilbert–Schmidt operators, we have shown that operator ideals and spectral decay fundamentally enlarge and refine the Bohr radius. The sharp characterization via the unilateral shift confirms the persistence of the classical bound in infinite-dimensional settings, while weighted Hilbert spaces and entropy methods reveal new asymptotic behaviors and geometric insights.

The unified framework presented here bridges classical analysis with modern functional analysis, recovering Bohr's original theorem, the Boas–Khavinson extension, and Defant's operator-theoretic generalization as special cases. In doing so, it establishes the Bohr radius as a central tool for understanding operator-valued analytic functions and their structural constraints.

Future directions include exploring analogues in operator-valued Hardy spaces, deeper connections with entropy geometry, and applications to quantum information and spectral theory. Ultimately, this work positions the Bohr radius as a unifying principle across analysis and operator theory, opening pathways for both theoretical advancement and practical application in infinite-dimensional functional analysis.

7 References

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